A TEXT-BOOK OF
GENERAL PHYSICS

(*FOR DEGREE COURSE*)

BY

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University Gold Medallist and Prizeman

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Humbly dedicated

to the living memory of

SIR ASUTOSH MOOKERJEE

K.T., C.S.I., M.A., D.L., D.SC.

List of Abbreviations

A. U.—Agra University
B. U.—Bombay University
C. U.—Calcutta University
D. U.—Delhi University
E. P. U.—East Punjab University
L. U.—Lucknow University
M. U.—Madras University
N. U.—Nagpur University
P. U.—Punjab University
R. U.—Rajputana University

PREFACE

TO THE FIRST EDITION

The present book is mainly based upon the newly formed B.A. and B.Sc. pass syllabus on General Physics of the Calcutta University. It is expected to meet the requirements of the B. A. and B.Sc. syllabuses of other Indian Universities as well. More advanced portions of the subject are printed in small type, which may be omitted in the first reading.

In preparing the book only rudimentary principles of Differential and Integral Calculus have been used where such use makes the subject comparatively short and intelligible and avoids the more lengthy algebraic treatment. To suit the students who have no knowledge of Calculus, algebraic treatments have been inserted in most places as alternatives to those involving the application of Calculus.

I take this opportunity to acknowledge my indebtedness to most of the standard books of reference on the subject available in the market, which I have fully consulted.

I express my thanks to my worthy colleagues, Prof. A.K. Banerjee and Prof. H. P. De, for their kind interest in the book and to the former in particular for going through the proof-sheets. Thanks are also due to my colleague, Prof. A. K. Sen, for preparing the index.

In conclusion, I express my gratitude to Prof. Jogesh Chandra Mukherjee. M. A. of the University College of Science, Calcutta, for going through portions of the manuscript and giving very helpful suggestions.

Physical Laboratory,
Vidyasagar College, Calcutta,
July, 1933.

T. M. M.

ERRATA

Page	line	for	read
28	9	M^m	M_m
136	8	R^{m3}	$R_m{}^3$
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GENERAL PHYSICS

CHAPTER I

FUNDAMENTAL PRINCIPLES

1. **Physical Quantity**—*A physical quantity* as distinct from a numerical or arithmetical quantity may be defined as a measurement of some physical property. Thus 1, 3, 10, etc. are numerical or arithmetical quantities whereas 1 foot, 3 lbs, 10 ft. per sec., etc., are physical quantities.

Mass, length, velocity, acceleration, etc., measure the physical properties of a body and are thus *physical quantities*. Length, mass and time are called the three fundamental physical quantities. Most of the other physical quantities are related to these. The units of length, mass and time are therefore called the *fundamental units* of measurement, The *dimensions of a physical quantity* represent an expression which shows the relation of the quantity to the fundamental physical quantities.*

2. **Scalar Quantity**—A quantity which involves a magnitude but no direction for its measurement, is called a *scalar* or *undirected* quantity. The mass or length of a body involves a magnitude only for its measurement and has no directional property and is thus a scalar quantity. These quantities may be compounded by direct algebraic addition. The combined mass of a number of bodies is the sum of the masses of the separate bodies.

3. **Vector Quantity**—A quantity which involves a magnitude as well as a direction for its measurement is called a *vector* or *directed* quantity. The displacement or velocity of a moving body involves a magnitude as well as a direction for its measurement and is thus a vector quantity. A vector quantity may be represented by a straight line whose length is a convenient measure of the magnitude of the vector and whose

---

*The dimensions of some physical quantities are given in Chapter XV.

direction and sense are the same as those of the vector. These quantities cannot, in general, be compounded by direct algebraic addition. Thus, the resultant of two separate displacements d_1 and d_2 in not d_1+d_2 except in the case when the displacements take place in the same direction.

Vector quantities can be compounded by the *parallelogram law*. If two vectors are represented in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of the of the parallelogram drawn through the point of intersection of the two sides The parallelogram of forces, velocities, etc., are only the special cases of this general law.

4. **Fundamental Units**—The fundamental units chosen are those of *length*, *mass* and *time*. Most of the other physical units are defined in terms of these units.

In the metric system, the standard of length is the *metre*. It is defined as the distance between the ends of a certain platinum bar kept at 0°*C*. It has been constructed by the International Bureau of Weights ane Measures and is called the International Prototype Metre. The *British yard* is defined as the distance between two transverse lines in the two gold plugs on a bronze bar maintained at 62°*F* and kept in the office of the Exchequer.

The French or metric standard of mass is a *kilogramme*. It is the mass of a lump of platinum made by Borda. It is as nearly as possible of the same mass as that of a cubic decimetre of distilled water at 4°*C*. The British standard mass is the *pound avoirdupois*. It is the mass of a piece of platinum kept in the office of the Exchequer.

The unit of time in either system is the *mean solar second*. It is the 86,400th part of the *mean solar day*. It is explained in detail in Art. 6.

5. **Mass and Weight**—The mass of a body is defined as the quantity of matter in it. This definition of mass is not of much practical importance as we are more concerned with the effects of mass than with its exact formulation. The most important effect which the mass of a body produces is its weight which is defined as the attraction the earth exerts on it. The mass of a body is thus a property peculiar to the body itself and is independent of neighbouring bodies and also of its position. The weight of a body, on the other hand, is due to earth's existence and depends upon the position of the body with respect to the earth.

The difference between weight and mass can also be illustrated from another stand-point :—the effect of the weight of a body is to cause it to move, while the effect of the mass of a body gives it a property of *inertia* or reluctance to movement (Art. 65).

The weight of a body is thus fundamentally different from its mass, yet exact experiments show that there is a connection between the two and that this connection is of direct proportionality. The first experimental truth of this was obtained by the experiments of Galileo in 1590 in which bodies of different masses were let fall from the leaning tower of Pisa. They reached the ground almost simultaneously thus proving equal accelerations to their motion. The truth was established with greater exactitude by the guinea-feather experiment of Newton.

The proportionality between mass and weight received a more exact confirmation from the pendulum experiments. If the value of the acceleration due to gravity (g) is deduced from the periods of oscillation of different pendulums having different masses, the values will be different if the weight of each pendulum was not in direct proportion to its mass. Newton and later Bessel in 1830 constructed pendulums with cavities in the bobs, and inserting materials of different densities ; no difference in the value of g was, however, detected greater than the possible limit of accuracy.

6. **Time**—Astronomers recognise three kinds of time—(1) *apparent solar time*, (2) *mean solar time* and (3) *side-real time*. (1) Apparent solar day—if at any place the gnomon or pin of the sun-dial is adjusted to lie in the north and south vertical plane, the interval between two successive transits of the shadow of the sun over the twelve o'clock mark is the apparent solar day. An apparent solar day is not constant throughout the year, it is longer in December than in September. So apparent solar day time is not kept in clocks and watches. (2) Mean solar day—the mean length of all apparent solar days in a solar year is called the mean solar day. It is a constant unit and is kept by all well-regulated clocks and watches. 1/86400th part of the mean solar day is called the *second*, the unit of time. (3) Sidereal day—the interval of time taken by the fixed stars to complete a revolution round the pole is called the sidereal day. 1/24th sidereal day is the sidereal hour. There are 365·2421 *mean solar* days and 366·2421 *sidereal* days in a solar year. The sidereal second is sounded in astronomical clocks and is thus slightly shorter than the second sounded in ordinary clocks which keep the mean solar time.

7. The Balance—The mass of a body is determined by 'weighing' the body in a common or beam-balance. From the fact that the mass and the weight of a body are in direct proportionality, it is evident that if at one place two bodies have the same weight, they must have the same mass.

The sensitiveness of a beam-balance for a particular load is defined as the deflection in scale-divisions produced when a weight of 1 milligramme is added to one scale-pan in excess to the load placed in each of the scale-pans. The factors which govern the sensitiveness of a balance may be seen from the following consideration :

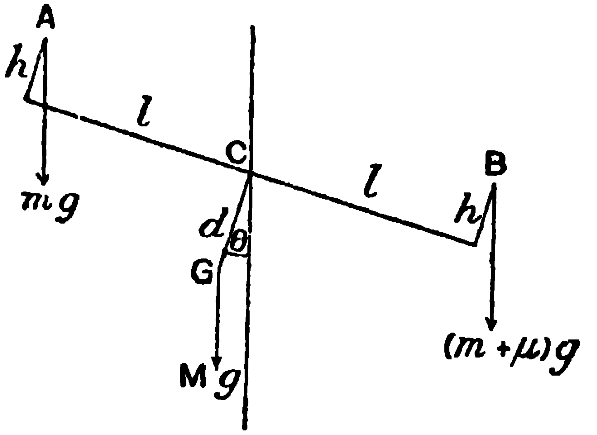

Fig. 1

Fig. 1 represents the vertical section of a balance through the centre of the beam cutting the knife-edges at A, B, C. Let l be the length of each arm of the balance ; d, the depth of its centre of gravity below C and h, the height of A and B above C when the balance is at rest in the horizontal position. Let also M represent the mass of the beam and m and $m+\mu$ represent the masses of the scale-pans with their loads where μ is very small. Because of the small difference in loads on the scale-pans the beam will come to rest inclined at a small angle θ to the horizontal.

Taking moments of the different weights about C,

$$Mgd \sin \theta + mg\,(l \cos \theta - h \sin \theta) = (m+\mu)g(l \cos \theta + h \sin \theta)$$

$$\text{or } (Md - 2mh - \mu h) \sin \theta = \mu l \cos \theta.$$

Since θ is small, $\sin \theta = \theta$ and $\cos \theta = 1$. Also neglecting the term $\mu h \theta$,

$$(Md - 2mh)\,\theta = \mu l$$

$$\therefore \quad \frac{\theta}{\mu} = \frac{l}{Md - 2mh}$$

If $\mu = 1$ milligram, θ measures the sensitiveness for the load placed on either scale-pan.

The above expression shows that when three knife-edges are co-planer, *i.e.*, when $h=0$, the sensitiveness is independent of the load. If the terminal knife-edges A and B are above the central knife-edge, *C*, the sensitiveness increases with load. When the terminal knife-edges are below the central knife-edge, the sensitiveness decreases with increase of load. Sensitiveness is also increased by increasing the length of the balance-beam and decreasing its mass, but the necessity of stiffness of the beam puts limits in fulfilling these demands. In very sensitive balances beams have lattice structure to combine lightness with stiffness. Also the sensitiveness increases as *d*, the distance of the centre of gravity from the central knife-edge, decreases. But this accompanies with loss of stability: for, the restoring moment being decreased, the balance will have a longer period of oscillation. In some balances, a weight can be adjusted along a vertical direction through central knife-edge whereby *d* may be varied and the balance made suitable for greater sensitiveness or stability.

8. Faults in a Balance—(a) *Arms of Unequal Lengths*—A body of unknown mass M to be weighed is first counterpoised in the right-hand scale-pan by a mass M_1 and then the body is placed on the left-hand scale-pan. A different mass M_2 will now counterpoise it. Let l_1 and l_2 be the lengths of the left-hand and right-hand balance-arms respectively. Taking moments,

$$M_1 g\, l_1 = Mg\, l_2 \ldots (1) \quad \text{Also } M_2\, g l_2 = Mg\, l_1 \ldots (2)$$

Multiplying (1) and (2), $M^2 = M_1\, M_2$ *or* $M = \sqrt{M_1\, M_2}$

Dividing (1) by (2), $\frac{M_1}{M_2} \cdot \frac{l_1}{l_2} = \frac{l_2}{l_1}$ $\quad . \quad \frac{l_1}{l_2} = \sqrt{\frac{M_2}{M_1}}$

(b) *Scale-pans of Unequal Masses*—This can be tested by taking pointer readings by interchanging the pans. If the pointer reading is altered, the pans are of unequal weights. The positions of the screws on the pans may be altered to make the balance even.

(c) *Knife-edges*—With use the knife-edges become rounded. This alters the line of contact with the support as the beam is tilted. The effect is to produce a small change in the length of the balance-arms.

(d) *Weights Inaccurate*—A set of weights may be tested among themselves, one of highest denomination being tested with lower ones. Thus, 20 gms. may be compared with 10+5+2+2+1 etc., and errors of the lower denominations are determined. The error of the highest one is determined by comparison with a weight known to be accurate.

9. The Laws of Motion—Newton in his *Principia* published in 1686, enunciated the three fundamental laws of motion. On them is based the whole of Dynamics and on Dynamics, the whole of Astronomy. The application of these laws enabled astronomers to predict the occurrences of eclipses with an accuracy which defies any suggestion that the laws are wrong. It is only when this Newtonian mechanics is applied to the motion of the constituents of the atom or to the motion of celestial bodies that small discrepancies occur. These laws are—

Law I.—Every body preserves its state of rest or of uniform motion in a straight line, except in so far as it is compelled to change that state by forces impressed on it.

Law II.—Change of motion is proportional to the moving force impressed, and takes place in the straight line in which the force acts.

Law III.—An action is always opposed by an equal reaction or the mutual actions of two bodies are always equal and act in opposite directions.

10. The first law states a fundamental property of material bodies called *inertia*, according to which a body has no innate tendency to alter its state of rest or of uniform motion in a straight line. The law may be regarded as a definition of force—*i.e.*, a force is anything which changes or tends to change the state of rest or of uniform motion of a body along a straight line.

11. To understand **the second law** it is necessary to know what Newton meant by the 'motion' of a body. 'Motion of a body', he states, 'is its quantity arising out of its mass and velocity conjointly'. The motion referred to is regarded as the *momentum*, *i.e.*, the product of mass and velocity of the body. 'The moving force impressed' in the law means the *impulse* of the moving force calculated over the time for which the change of momentum is considered. The impulse of the force over a certain time is measured by the product of the force and the time.

Suppose, m is the mass of a body moving with a velocity v. Let a force of magnitude P act upon the body for a time t and alter its velocity from v to v'. Therefore, the change of momentum of the body in the time t is $mv' - mv$. Hence, according to the law

$$Pt \propto (mv' - mv)$$

$$\text{or, } P \propto \frac{mv' - mv}{t} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

or, $P \propto$ rate of change of momentum.*

From (1), $P \propto m \times$ rate of change of velocity

or, $P \propto mf$ (where f is the acceleration produced).

Let $P = kmf$ ... ... (2)

(where k is a constant)

Let a force be so chosen that when applied to unit mass, it produces unit acceleration in the mass and call the force as unit force.

Hence, when $m=1$, and $f=1$, we have $P=1$ $\therefore$ $k=1$

Thus, the relation (2) reduces to

$P=mf$ ... ... (3)

Relation (3) specifies the *unit* for the measurement of force. The unit force is that force which acting on a unit mass produces in it the unit of acceleration.

In the metric (C. G. S.) system, the unit of mass is one gramme and the unit of acceleration is one centimetre per second per second ; the unit of force is called a '*dyne*'. In the British (F.P.S.) system, the unit of mass is one pound and the unit of acceleration is one foot per second per second ; the unit of force is called a '*poundal*'. These units are called the absolute units.

Besides the above units, the weight of unit mass has also been chosen as a unit of force, such a unit is called the gravitational unit. Since the weight of a mass varies from place to place on earth's surface, such a unit of force is different at different places. To maintain the fixity of the unit the weight of unit mass at sea-level and at latitude 45° is chosen as the unit of force.

In the metric gravitational system. the unit of force is the weight of one gramme at sea-level and at latitude 45°, This unit is known as one gramme-weight (1 gm. wt.).

1 gm. wt.$=g$ dynes, where $g=980{\cdot}6$ cm./sec^2 at the aforesaid place.

In the British gravitational system, the unit of force is the weight of one pound at sea-level and at latitude 45°. This unit is known as one pound-weight (1 lb. wt.).

1 lb. wt.$=g$ poundals, where $g=32{\cdot}17$ ft/sec^2 at the aforesaid place. Thus, the first part of the second law provides a method for the measurement of force.

The latter part of the second law states that the change of motion takes place in the straight line in which the force acts.

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*From this, the second law is otherwise stated as—"The rate of change of momentum is proportional to the impressed force and takes place in the straight line in which the force acts".

Suppose a particle is moving in a direction AB and a force is impressed upon it in another direction AC. Then according to the law, the velocity of the particle in the direction AB will be unaffected by the force which will only generate an acceleration on the particle in the direction AC. If we want to find the real velocity of the particle at the end of the unit time since the force commenced to act, we shall have to compound the velocity of the particle along AB with the velocity generated in the particle in the unit time by the force along AC.

Similarly, if there be several forces acting upon a body in various directions, to find their combined effect, first calculate the effect produced by each force supposing that the others did not exist and then compound all these effects. The principle is known as the **Physical Independence of Forces.**

12. The third law of motion is a generalization of everyday phenomena. It states that forces in nature always occur in pairs, —one of which is called the action and the other the reaction. If a book is kept on a table the book presses the table with a force equal to its weight. This force may be called the *action.* The table exerts an equal and opposite force on the book. This force is the *reaction* of the table on the book.

When a body falls under the action of gravity, the earth exerts a force of attraction on the body, which may be called the action. The body exerts an equal force of attraction on the earth which is the reaction. Because of the huge mass of the earth compared to that of the body, the acceleration with which the earth moves is inconceivably small compared to the acceleration with which the body moves. The centre of mass of the system, however, remains unaltered.

A direct corollary of the third law of motion is the law of **Conservation of Momentum.**

Let two bodies, A and B, either of which is uninfluenced by any third body, move in any manner. According to the third law of motion, the force which A exerts upon B is equal and opposite to the force which B exerts on A. Since a force is measured by the rate of change of momentum produced by it, it follows that the rate of change of momentum of B is equal and opposite to the rate of change of momentum of A. Calculated over a definite interval of time, the change of momentum of B is equal and opposite to that of A. Thus, the sum total of the changes of momentum of A and B taken with proper sign is zero. In other words, the algebraic sum of the momenta of the two bodies is a constant. The law can be extended to any number of bodies.

Examples

1. A body of mass 3 lbs is falling under gravity at the rate of 100 feet per second. What is the uniform force that will stop it (*a*) in 2 seconds, (*b*) in 2 feet ?

The body moves with an acceleration (acceleration of gravity) under its weight. So an opposing force of 3 lbs. wt. will counteract the acceleration and it will move with a uniform velocity of 100 feet per second according to the first law of motion.

(*a*) Let an additional opposing force F producing a retardation f bring it to rest in 2 seconds. From the equation

$$v=u+ft, \quad 0=100-2f$$

$\therefore \quad f=50$ ft. per sec. per sec.

By second law of motion the additional opposing force

$F=3\times 50$ or, 150 poundals$=\frac{150}{32}$ or $4\frac{11}{16}$ lbs wt.

Hence the total opposing force to be applied

$=3+4\frac{11}{16}$ or, $7\frac{11}{16}$ or 7·7 lbs wt. nearly.

(*b*) Let the additional opposing force needed to stop it in 2 feet be F' and let it produce a retardation f'. From equation

$$v^2=u^2+2f's, \quad 0=(100)^2-2\times 2f'$$

$\therefore \quad f'=2500$ feet per sec. per sec.

By second law of motion this additional opposing force

$F'=3\times 2500$ or, 7500 poundals$=\frac{7500}{32}$ or, $234\frac{3}{8}$ lbs. wt.

Hence the total opposing force to be applied

$=3+234\frac{3}{8}$ or, $237\frac{3}{8}$ or, 237·375 lbs wt.

2. A shot of mass m is fired from a gun of mass M with a velocity u relative to the gun ; find the actual velocities of the shot and the gun.

Let u_s and u_g be the actual volocities of the shot and the gun respectively since the velocity of the gun is opposite to that of the shot, the sum of their actual velocities is u.

$$\therefore \quad u_s+u_g=u \quad \ldots \quad \ldots \quad (1)$$

Also from the principle of conservation of momentum

$$mu_s=Mu_g \quad \ldots \quad \ldots \quad (2)$$

Solving the equation (1) & (2)

$$u_s=\frac{Mu}{M+m} \text{ and } u_g=\frac{mu}{M+m}$$

3. A man whose mass is 8 stone, stands on a lift which moves with a uniform acceleration of 12 ft.-sec. units. Find the reaction of the floor when the lift is (*a*) ascending and (*b*) descending.

(*a*) Let R be the reaction of the floor (acting upwards) in poundals. The resultant upward force on the man

=Reaction of the floor − wt. of the man

$= R - 8 \times 14 \times 32$ poundals.

$\therefore$ From 2nd law of motion

$R - 8 \times 14 \times 32 = 8 \times 14 \times 12 \quad (P = mf)$

$\therefore \quad R = 8 \times 14 \times 44$ poundals or, 154 lbs. wt.

(*b*) Here the resultant downward force on the man

=wt. of the man − Reaction of the floor

$= 8 \times 14 \times 32 - R$ poundals

From 2nd law of motion

$8 \times 14 \times 32 - R = 8 \times 14 \times 12 \quad (P = mf)$

$\therefore \quad R = 8 \times 14 \times 20$ poundals or 70 lbs. wt.

If the lift is at rest or it moves with uniform velocity, the reaction of the floor is equal to the weight of the man as in this case $f = 0$.

CHAPTER II

WORK, POWER, ENERGY AND FRICTION

13. Work done by a Force—Work is said to be done *by* a force when the point of application of the force moves in the direction of the force. When a body falls freely by the attractive force of the earth, work is said to be done *by* gravity.

Work is said to be done *against* a force, when the point of application of the force moves opposite to the direction in which the force acts. When a body is raised vertically upwards (by some external agency) work is said to be done *against* gravity (by the external agency).

14. Measurement of Work—Work is measured by the product of the force and the distance through which the point of application of the force moves in the direction of the force.

Thus, when a body P initially at A moves under the action of a force F, from A to B along the direction of the force (fig. 2), work done by the force $= F \times AB$. When, on the other hand, the body moves from A to C (fig. 2) opposite to the direction of the force, work done against the force $= F \times AC$.

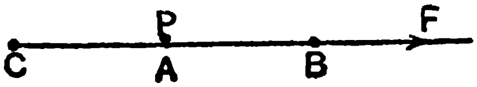

Fig. 2

15. If the direction of the force and line of motion of the body be inclined to each other, as in the case of a body moving along an inclined plane under the action of gravity, the work will be measured by the product of the force and the component of the displacement of the body along the direction of the force.

In fig. 3, a force F is applied along the line AC to a body initially at A. Suppose the body moves along the line AB inclined at an angle θ to AC through a distance AD. Draw DE perpendicular to AC, AE represents the component of AD along AC.

$$\text{The work done} = F \times AE$$

$$= F \times AD \cos \theta \quad \ldots \quad \ldots (1)$$

Since the displacement AD is equivalent to two displacements AE and ED, it is clear that no work is done for a displacement ED of the body which is at right angles to the direction of the force.

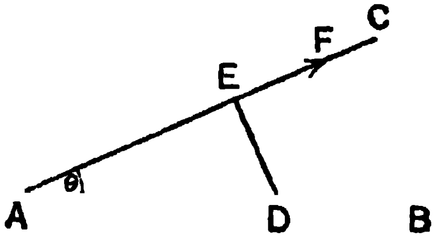

Fig. 3

The work done in the expression (1) above, can also be put as $F \cos\theta \times AD$. Again, $F \cos\theta$ measures the component of the force in the direction of the displacement. Thus, the work can also be measured by the product of the displacement and the component of the force in the direction of the displacement. The other component $F \sin\theta$ of the force perpendicular to the direction of displacement does not contribute to the measure of the work.

Since work is measured by the product of the force and the displacement it makes along the direction of the force, force may be defined as the *space rate* at which work is done.*

16. Work done by a Couple—Let a rigid body be acted upon by a system of foces of which F represents a typical one acting at a point A of the body (fig. 4).

Suppose this system causes the body to rotate through a small angle $\delta\theta$ about an axis perpendicular to the plane of the paper through O such that A moves to A_1. Since $\delta\theta$ is small $A\ A_1$ may be regarded straight.

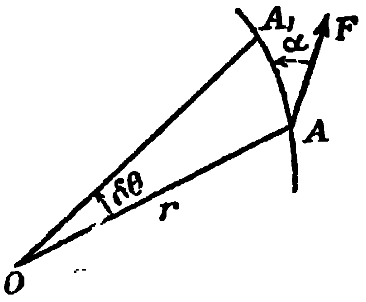

Fig. 4.

If the angle between AA_1 and the direction of F is a, the work done by the force F

$$=AA_1 \times F \cos a \text{ (Art. 15)}$$
$$=r\delta\theta \times \text{component of } F \text{ along } AA_1$$
$$=\delta\theta \times \text{moment of } F \text{ about the axis of rotation.}$$

If the moment of all the forces in the system about O be c, then the work done by the system is $c\ \delta\theta$. In particular if the forces form a couple, the work done is equal to the product of the moment of the couple and the angle of rotation.

17. ✓Units of Work—The *C. G. S.* unit of work is the *dyne-centimetre* or *erg*. This is the work done by a force of one dyne when its point of application moves through one centimetre in

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*Conversely, work may be defined as the *space integral* of a force.

the direction of the force. For practical purposes this unit is small and work is measured in terms of '*Joule.*'

$$\text{One joule} = 10^7 \text{ ergs.}$$

The *F. P. S.* unit of work is the *foot-poundal.* This is the work done by a force of one poundal when its point of application moves through one foot in the direction of the force.

In the gravitational system, the unit of work is the work done by a force equal to the weight of unit mass when its point of application moves through unit distance in the direction of the force.

In the metric gravitational system, the unit of work is the work done by a force equal to the weight of one gramme when its point of application moves one centimetre along the direction of the force. The unit is known as *gramme-centimetre* (gm. cm.). From Art. 11.

$$1 \text{ gm. cm.} = g \text{ ergs, where } g = 980{\cdot}6 \text{ cm./sec.}^2$$

In the British gravitational system, the unit of work is the work done by a force equal to the weight of one pound when its point of application moves one foot along the direction of the force. The unit is known as *foot-pound* (ft. lb.). From Art. 11.

$$1 \text{ ft. lb.} = g \text{ foot-poundals, where } g = 32{\cdot}17 \text{ ft./sec.}^2$$

18. Power—The power or activity of an agent doing work is the rate at which the work is done. Thus, power may be considered as the *time rate* of work.[*]

In the *C. G. S.* system, the unit of power is *one erg per second* and the practical unit of power is *one joule per second* which is called one *watt.* Hence,

$$1 \text{ watt} = 10^7 \text{ ergs per second.}$$

In the *F. P. S.* system, the unit of power is one *foot-poundal per second.* For practical purposes the engineers use a unit called *horse power.* (*H. P.*)

$$\begin{aligned} 1\ H.\ P. &= 550 \text{ foot-pounds per second} \\ &= 550 \times 32{\cdot}17 \text{ or, } 17693{\cdot}5 \text{ foot-poundals per second.} \end{aligned}$$

Again, 1 *H. P.* = 746 watts.

19. Energy—Energy of a body is its capacity for doing work. A body is capable of doing work under two circumstances :—(1) due to its motion and (2) due to its position or configuration.

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[*] Conversely, work may be regarded as the *time integral* of power (compare Art. 15).

(1) When a bullet strikes a tree, it penetrates through the tree and does work against the resisting force offered by the tree until it comes to rest. This energy of the body is called **Kinetic.** (2) When a body is raised above the surface of the earth, it can do work during its fall to the surface of the earth ; for, it can raise a weight smaller than its own which is tied to it by a string passing over a pulley. This energy of the body is called **Potential.** It is the gravitational potential energy of the body. A bent spring or a piece of stretched india-rubber cord is possessed of elastic potential energy.

20. Measurement of Energy—The kinetic energy of a body is measured by the work the body can do against an opposing force before it comes to rest. Let m be the mass of a body moving with a velocity v, and suppose by an opposing force F, the body is brought to rest after it has travelled a distance s from the instant the force is applied. Therefore, the work done by the body against the opposing force$=F\times s$.

According to the second law of motion, the effect of the opposing force F on the body is to produce acceleration of magnitude F/m opposite to its direction of motion. Thus, the distance s travelled by the body before coming to rest, will be given by

$$0=v^2-2\frac{F}{m} \qquad (\text{cf. } v^2=u^2+2fs)$$

$$\text{or, } s=\frac{1}{2F}mv^2$$

Hence, the work done by the body before coming to rest or its kinetic energy

$$=F\times s=\tfrac{1}{2}mv^2=\tfrac{1}{2}\times \text{mass}\times(\text{velocity})^2.$$

The potential energy of a body is measured by the work which the body can do in passing from its present position or configuration to a standard position or configuration known as zero position. The zero position of a body raised to a height above the earth's surface is the surface of the earth (sea-level), for a bent spring it is its unbent position, and for a stretched india-rubber cord it is its unstretched position.

If a body of mass m is raised to a height h above the earth's surface the work which it can do in falling to the surface of the earth is evidently mgh which is its gravitational potential energy.

To calculate the elastic potential energy of the bent spring or of the stretched india-rubber cord we shall have to know the elastic forces called into play.*

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* **See Chapter IX**

21. Other forms of Energy—By heat water is converted into steam and the steam in steam-engine is used to drive the piston and do work thereby. An electrified body can attract uncharged bodies towards itself and do work. Light when absorbed by a body causes a rise in its temperature. The propagation of sound through a medium is associated with vibratory motion of the molecules of the medium. Thus, Heat, Light, Sound, etc., are the different forms of energy. Energy is also associated with chemical reactions. Thus, heat is evolved when water mixes with lime.

22. Conservative System of Forces—Consider that a body of mass m is raised through a height h. The work done *against* gravity$=mgh$. If the body is allowed to return to its original position the work done *by* gravity$=mgh$, this work is available from the body. The net work done upon the body is zero.

Next consider a body is pulled through a distance S against a frictional force F, the work done by the agency is FS. If the body is brought to the original position again an equal quantity of work FS will be done by the agency so that the total work done upon the body is $2FS$.

The essential difference between the two is that in the former, work done can be recovered from the system during its return to the original position but in the latter no recovery of work done is possible.

When the forces acting on a system of bodies are of such a nature that the algebraic work done in performing any series of displacements, whereby the original configuration is retained is zero, then these forces are called a conservative system of forces. Gravitational, electrostatic forces are conservative whereas frictional, viscous forces are non-conservative.

It can be shown that the work done by a conservative system of forces in moving a body from one configuration to another is independent of the path traversed by the body. Let the work done in taking a body from A to B along a path (1) be W_1 and along a path (2) be W_2 ; while the work done in passing from B to A along a path (3) be W_3. Then from the definition of conservative forces

$$W_1+W_3=0 \qquad \text{and} \qquad W_2+W_3=0$$

$\therefore$ $W_1=W_2$ or the work done is independent of the path traversed by the body.

23. Conservation of Energy—The principle of conservation of energy states that the total energy possessed by an *isolated* body or a system of bodies remains a constant. It may change from one form to another or pass from one body to another inside the system without any loss or gain. In practice, it is not always easy to isolate a body so that no energy can leave it or get into it from its surroundings, but the most careful measurements have as yet shown no exceptions to this principle. Machines devised to run with perpetual motion without any supply of energy, in defiance of this principle, have not lived up to the hopes of their inventors. The principle of conservation of energy stated by Clerk Maxwell runs as follows—"The total energy of any material system can neither be increased nor diminished by any action between the parts of the system, though, it may be transformed into any of the forms of which energy is susceptible."*

24. **Availability of Energy**—When a certain amount of energy in one form disappears, an equivalent amount energy in other form makes its appearance ; thus, the total quantity of energy of the universe remains constant. But to us there is a very important difference between these forms of energy. By this transformation, the energy which is available to us for useful purposes is gradually decreasing ; for example, when a stone falls on the ground its kinetic energy is converted into heat energy on reaching the ground. The heat thus produced is diffused into surrounding objects. Energy in the form of uniformly diffused heat is unavailable to us. We can only utilise the heat energy of a body which is hotter than surrounding objects for useful purposes.

This continual decrease of the useful form of the energy which accompanies every phenomenon signifies the arrival of a time when the universe will be a uniformly hot inert mass and its energy will be unavailable to us.

25. **Friction**—If a block of wood is set in motion in a straight line on a table, although we might except that it would go on moving due to its inertia, it is found to come to rest after it has travelled a short distance.

If to a body lying on a table a force is applied, it is found that no visible motion of the body takes place until the force exceeds a certain magnitude depending upon the nature of the surfaces of the body and the table. Further, when actual motion of the body takes place under a certain force, we would expect that the acceleration of the body will be given by the applied force divided by the mass of the moving body. But the acceleration of the body is found to be considerably less. These facts suggest that a resistance is offered to the motion of one body over another with which it is in contact. This property of the bodies of offering resistance, when two bodies in contact are moving relative to each other, is called *friction* and the resisting force is called the *force of friction* or simply *friction* between the two bodies.

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*It is not quite certain that the principle of conservation of energy holds good with reference to processes that take place within an atom.

The magnitude of the force of friction between any two bodies in contact varies differently depending upon the condition of the two bodies with reference to their state of rest or of relative motion.

26. (*a*) **Statical Friction**—Consider two bodies in contact with one another. Let a force be applied to one of the bodies tending to make it move on the surface of the other. If the applied force is too small, no motion of the body takes place. The friction called into play is said to be *statical*. The direction and magnitude of the frictional force are given by the following law :

Law I.—The direction and magnitude of the frictional force are just sufficient to preserve equilibrium.

Hence, the direction of the frictional force is opposite to the direction of the applied force and the magnitude of the frictional force is the same as that of the applied force.

(*b*) **Limiting Friction**—When the applied force on one body is gradually increased to such a magnitude that the body is just on the point of moving on the other, the equilibrium is called *limiting* and the friction called into play is called the *limiting friction.*

The direction of limiting friction is opposite to the direction in which the body tends to move. The magnitude of the limiting friction is given by the following two laws :

Law II.—The limiting friction between two surfaces is proportional to the normal reaction between them.

Law III.—The limiting friction is independent of the area of the surfaces in contact.

(*c*) **Kinetic Friction**—When the applied force is further increased, the frictional force is no longer able to retain the equilibrium of the body. The friction called into play is called the *kinetic friction.* The following law relates qualitatively to the magnitude of the frictional force, the direction of which is opposite to the direction of motion :

Law IV.—When motion ensues, the frictional force is less than the limiting friction and is independent of the velocity and area of the surfaces in contact.

27. Verification of the Laws of Limiting Friction: Coefficient of Friction—The laws of limiting friction admit of direct experimental verification in the following way: To verify Law II, take a large smooth sheet of wood AB and fix it firmly on a table in a horizontal position (fig. 5). Take a second smooth piece of wood C to act as a sliding piece. Place it on AB, attach a light string to C and pass the string over a smooth pulley fixed at the end of AB so that the portion of the string from the wood to the pulley remains in horizontal position. To the other end of the string attach a scale-pan.

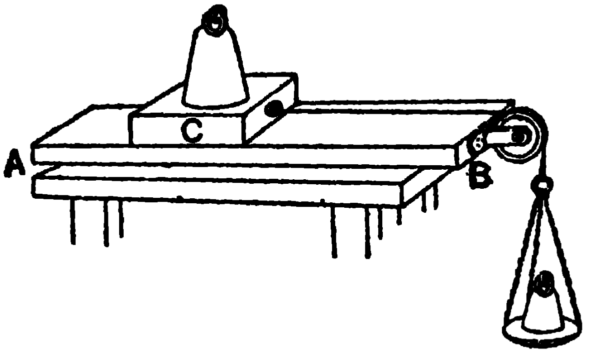

Fig. 5

Place a load on C and put weights on the scale-pan commencing from a small weight and increasing it gradually until C just moves on AB. The friction is now a limiting one. Its magnitude is practically the weight of the scale-pan and the weights placed on it. Let the weight of the scale-pan together with the weights placed on it be W. If the weight of the sliding piece C together with that of the load placed on it is W', the normal reaction of AB, which is equal to the weight of the sliding piece and that of the load acting vertically upwards, is W'. Calculate the ratio W/W'. Now put a different load on C and adjust the weight on the scale-pan by increasing it gradually from a small value until the sliding piece is again at the point of moving. Again calculate the ratio. It will be found that the ratio is a fairly constant quantity. This constant ratio of limiting friction to the normal reaction is called the *coefficient of friction*. It is denoted by μ.

Law III is verified by removing a portion of the sliding piece C and performing the experiment as before. It will be found that the value of μ remains the same. Thus, the limiting friction is independent of the area of contact.

28. Coefficient of Kinetic Friction—It has already been said that when motion ensues, the friction on the moving body is less than the limiting friction. The ratio of the force of friction on the body to the normal reaction of the plane is known as the *coefficient of kinetic* or *sliding friction*. Hence, if F represents the force of friction and R the normal reaction,

$\frac{F}{R}=\mu'$, where μ' is the coefficient of kinetic friction.

$$\therefore \quad F=\mu' R$$

Thus, the effect of kinetic friction on the body is to oppose its motion with a constant force $\mu' R$.

If a body of mass m moves on a horizontal plane under the action of a force P acting in the horizontal direction as in fig. 5, the normal reaction of the plane is mg and the force of friction is $\mu' mg$. Thus, the effective moving force on the body is $P - \mu' mg$. The acceleration of the body, by the second law of motion,

$$= \frac{P - \mu' mg}{m}.$$

If the surfaces of contact were *perfectly smooth*, no frictional force would exist and the acceleration of the body would have been $\frac{P}{m}$.

29. Angle of Friction—When the friction is limiting, if the frictional force and the normal reaction be compounded into a single force, this single force is called the *resultant reaction.* The angle which this resultant reaction makes with the normal reaction is called the *angle of friction* or somtimes the *limiting angle.*

Let A (fig. 6) be the point of contact of a body with a surface. The body is on the verge of moving rightward. Let N be the reaction of the surface on the body perpendicular to the surface represented by AB. By the law of limiting friction the frictional force will be of magnitude μN acting leftward, where μ is the coefficient of friction between the body and the surface. This force is represented by AC. Let AD represent the resultant R of these two forces. The angle which AD makes with AB is called the *angle of friction*. If λ represents the angle of friction, we evidently have

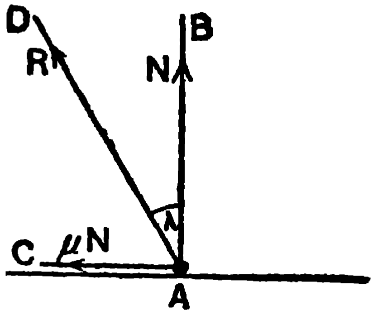

Fig. 6

$$R \sin \lambda = \mu N \quad \ldots \quad \ldots \quad (1)$$
$$\text{and } R \cos \lambda = N \quad \ldots \quad \ldots \quad (2)$$

Dividing (1) by (2), $\tan \lambda = \mu$

$\therefore$ *The tangent of the angle of friction is equal to the coefficient of friction.*

30. Angle of Repose—If a body rests on a rough plane and if the inclination of the plane to horizon is gradually increased, the condition of rest of the body will be a limiting one when the inclination becomes equal to the angle of friction.

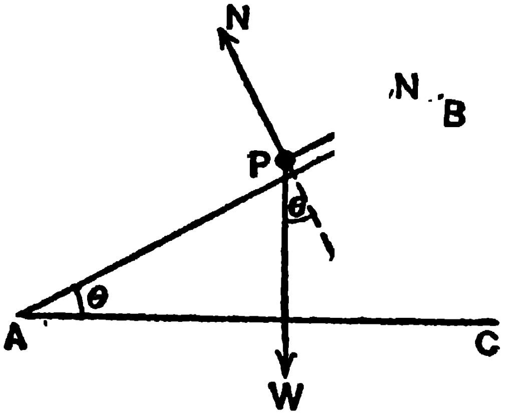

Fig. 7

Let P be a body (fig. 7) resting on the plane AB, and that the equilibrium of P is limiting when AB is inclined to the horizon AC at an angle θ. The weight of the body W acts vertically downwards, The component of W, perpendicular to the plane, balances the normal reaction N of the plane.

$$\therefore \quad W \cos \theta = N \quad \ldots \quad (1)$$

Also the component of W parallel to the plane, balances the force of friction μN.

$$\therefore \quad W \sin \theta = \mu N \quad (2)$$

Dividing (2) by (1) $\tan \theta = \mu$.

Again $\mu = \tan \lambda$, where λ is the angle of friction.

Hence, $\theta = \lambda$.

Therefore, the body rests until the inclination of the plane is equal to the angle of friction. This maximum inclination of the plane to the horizon without the body sliding, which is equal to the angle of friction, is called the *angle of repose.*

31. Rolling Friction—When a body such as a wheel rolls on a surface, the resistance called into play is known as the *rolling friction.* The rolling friction is the less the greater the diameter of the wheel. The rolling friction is of much less importance compared to the sliding friction at the axle of the wheel. Hence, the use of ball bearings at the axle.

32. Friction and Lubrication—The action of minimising friction is known as *lubrication* and the substances used for the purpose are called *lubricants*. A rational theory of friction and lubrication seems to have been first given by Osborne Reynolds who gave an interpretation of lubrication in terms of the theory of viscosity. He showed that in the case of journal bearings supplied with ample lubricant, the two solid surfaces are separated by a fluid of appreciable thickness ; such films are maintained and are able to support the pressure imposed on them by the relative motion of the parts. The theory has been found to be of great service in suggesting very efficient bearings and practically all modern forms which operate with sliding contact are designed on the basis of Reynolds' theory.

A more recent study of friction and lubrication has been made by Hardy in 1920. He finds that perfectly smooth and clean surfaces do not slide over one another due to the action of cohesive forces. The action of a lubricant thus seems to vary according as the solid surfaces are or are not fully separated by a layer of the lubricant. When the lubricating layer is of considerable thickness, the friction is called the *floatation friction*, and Reynolds observes that the frictional resistance depends on the viscosity of the lubricant and is independent of the area of contact and the relative velocity of the surfaces. When, however, the thickness of the lubricant is of the order of the molecular diameter, the frictional resistance depends upon the physical properties of the solid interfaces as well as on the boundary conditions. This type of friction has been termed by Hardy as *boundary friction.*

The present view of friction is that, friction is due to molecular attraction which one solid exerts on the other across the interface when that solids are perfectly clean, and to the intermolecular attraction in the lubricant when the solid surfaces are separated by a film. A layer of the lubricant behaves under a tangential stress like a solid. The molecules of the solid face exert different attractive forces on the various parts of the molecules in the lubricant causing them to be differently oriented. These forces will be greatest at the solid faces and least midway between them. Hence, the layer of the lubricant, though homogeneous in a direction parallel to the solid faces, is extremely heterogeneous in a direction normal to them. The frictional resistance tends to rotate the oriented molecules of the lubricant. With increase of shear, there is a break in the lubricant along a tangent plane ; the magnitude of the maximum tangential stress depends on the nature of the solid surfaces, the nature of the lubricant and on the type of orientation of the molecules of the lubricant.

Examples

1. A solid hemisphere of weight 12 lbs. and radius 1 foot rests with its flat surface on a table. How many foot-lbs. of work are required to turn it over so that it may rest with its curved surface in contact with the table.

The centre of gravity of a solid hemisphere is at a distance $\frac{3}{8}r$ from the centre of the plane surface on the radius perpendicular to it, where r is the radius of the hemisphere.

In the first position of the hemisphere it. C.G. its $\frac{3}{8}$ ft. above the surface of the table (as $r=1$ft). In the second position the C.G. is $\frac{3}{8}$ ft. below the centre of the hemisphere or $\frac{5}{8}$ ft. above the

surface of the table. Hence, to turn it from the first to the second position its C.G. is raised through $\frac{5}{8}-\frac{3}{8}$ or $\frac{1}{4}$ft.
Work required $=$ Weight of the body $\times$ height through which its C.G. is raised $=12\times\frac{1}{4}$ or 3 ft-lbs.

2. Find the H P. of an engine which can travel at the rate of 25 miles per hour up an incline of 1 in 100, the mass of the engine and load being 10 tons and the resistance due to friction, etc., being 10 lbs. wt. per ton.

In travelling 100 ft the engine rises a vertical height of 1 ft. Hence if θ be the inclination of the track to the horizontal,

$$\sin\theta=\frac{1}{100}$$

The component of the weight W of the engine down the incline $=W\cos(90-\theta)=W\sin\theta=10\times\frac{1}{100}$ or $\frac{1}{10}$ ton wt. or 224 lbs. wt.
The resistance due to friction $=10\times10$ or 100 lbs. wt.
The total force against which the engine moves
$=224+100$ or 324 lbs. wt.

The speed of the engine $=25$ miles per hour or $\frac{110}{3}$ ft per sec.
Hence the rate of doing work
$=324\times\frac{110}{3}$ or 11880 ft-lbs. per second.

Its H. P. $=\frac{11880}{550}$ or 21·6.

3. A mass of 4 cwt. falls from a height of 10 ft. upon an inelastic pile of mass 12 cwt, supposing the mean resistance of penetration to the ground to be $1\frac{1}{2}$ tons wt.; determine the distance through which the pile is driven to the ground.

Let u be the velocity of hammer on hitting the pile, then

$$u^2=2\times32\times10 \text{ or } u=8\sqrt{10} \text{ ft. per sec.} \quad \ldots \quad (1)$$

Let v be the velocity of hammer and pile just after impact. Then from the principle of conservation of momentum

$$(4+12)\,v=4u \text{ or } 16\,v=32\sqrt{10} \text{ from } (1)$$

$$\therefore \quad v=2\sqrt{10} \text{ ft/sec} \quad \ldots \quad \ldots \quad (2)$$

The resistance to penetration to the ground $=$ Resistance offered by the ground $-$ wt. of the mass and the pile

$$=(1\tfrac{1}{2}\times20\times112-16\times112) \text{ lbs. wt.}=1568\times32 \text{ poundals}$$

If d be the distance of penetration, we have from the principle of conservation of energy, the K. E. of the hammer and pile just after impact is equal to the work done by the resistance to penetration

$$\therefore \quad \tfrac{1}{2}\times16\times112\times40=1568\times32\times d \quad (v^2=40 \text{ from } (2))$$

$$\therefore \quad d=\frac{16\times112\times40}{2\times1568\times32} \text{ or } \frac{5}{7} \text{ ft.}=8\text{·}6 \text{ inches nearly.}$$

Exercise

1. What is meant by the conservation of energy ? It has been said that practically all the energy available on the earth has been derived from the sun. Justify this statement in the case of the light energy given out by an ordinary gas lamp, an electric glow lamp and a lightning flash respectively. (C. U. 1909)

2. A solid cone of weight W whose height is h and radius of base r stands on a horizontal plane. Find the work expended in upsetting it. (C. U. 1910)

3. Explain the following terms :—Static, kinetic, and rolling frictions ; limiting angle and angle of repose. How would you determine the coefficient of kinetic friction between a block of wood and table ? How does the kinetic friction affect the acceleration of the block when it is acted on by a force parallel to the table ? (C. U. 1919)

4. Write a short essay on the principle of conservation of energy. (C. U. 1926)

CHAPTER III

UNIFORM CIRCULAR MOTION

33. An important case of *curvilinear motion* or the motion in a curved path is a uniform circular motion. The motion of the earth round the sun takes place in an approximately circular orbit. The speed of a body moving uniformly in a circle is constant as it passes over equal arcs of the circumference in equal times. But its direction of motion changes continually and therefore, a force must be continually acting upon the body. The direction and magnitude of the force acting upon the body can be found in the following way :

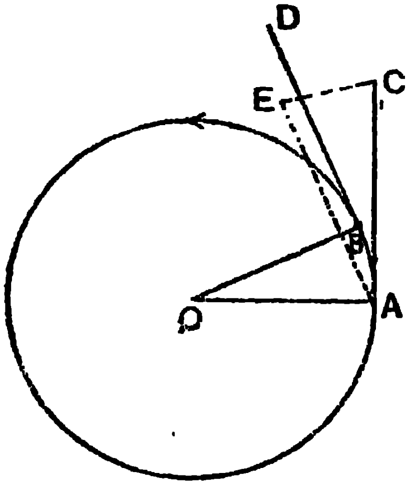

Fig. 8.

Let a particle of mass m move in a circle of centre O (fig. 8) and radius r with a uniform speed v. Consider two positions of the particle, A and B on its circular orbit very close to each other so that the arc AB approximates to a straight line. Let the particle take a time t to travel this arc.

The direction of motion of the particle at any point will be given by the tangent drawn to the circle at that point. The two tangents AC and BD to the circle, equal in length, represent the velocities of the moving particle at the two points A and B respectively in direction and magnitude. From A draw a straight line AE parallel to BD making $AE=BD$. Then AE represents the velocity of the particle at B in direction and magnitude. Join CE.

By the law of triangle of velocities, the velocity represented by AE is equivalent to two velocities represented in magnitude and direction by AC and CE taken in order. Thus, in moving from A to B the change of velocity of the particle is represented by CE.

Again, AC is normal to OA and AE is normal to OB ; therefore; the angles AOB and CAE are equal. Further, the triangles

AOB and CAE are isosceles and hence they are similar triangles.

$$\therefore \quad \frac{CE}{AB}=\frac{AC}{OA}=\frac{v}{r}$$

As the particle moves through AB in time t, $AB=vt$

$$\therefore \quad CE=\frac{v^2t}{r}$$

CE is the change of velocity in the time t.

Hence, the rate of change of velocity or the acceleration of the moving particle

$$=\frac{CE}{t}=\frac{v^2t}{rt}=\frac{v^2}{r}$$

If ω be the angular velocity of the particle, $v=\omega r$. Hence, the acceleration of the particle can also be represented by

$$\frac{v^2}{r} \text{ or, } r\omega^2$$

The direction of the acceleration at A is obtained by making t indefinitely small when AE becomes more and more nearly parallel to AC and ultimately coincident with it ; consequently CE in the limit becomes perpendicular to AC and hence, parallel to AO. Thus, the acceleration of the moving particle at any point is of magnitude $\frac{v^2}{r}$ or $r\omega^2$ directed along the radius through that point towards the centre of the circular path.

34. Centripetal and Centrifugal Forces—The fact that a particle moving uniformly in a circular orbit possesses an acceleration directed towards the centre of its circular path implies the existence of a force in the same direction which produces this acceleration. If m be the mass of the particle, the force acting on it is $\frac{mv^2}{r}$ or, $mr\omega^2$. This force is exerted by some agent which directs the particle towards the centre and is known as the **Centripetal force.*** The corresponding acceleration *of the particle* is called the *centripetal acceleration.* By the third law of motion an equal and opposite force is exerted by the particle on the agent. This latter force is called the **Centrifugal force** † The corresponding acceleration *of the agent* is called the *centrifugal acceleration.*

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*Latin 'centrum', centre and 'petere', to move towards.

† Latin 'centrum,' centre, and 'fugere', to flee.

35. Examples of Centripetal and Centrifugal Forces—If a ball tied to one end of a string is whirled round, the other end of the string being held by the hand, the hand provides the centripetal force which appears as the tension of the string. This prevents the ball from flying away along the tangent to its circular path. The ball exerts the centrifugal force cn the hand.

In the case of the rotation of the earth round the sun, the centripetal force is provided for by the attraction of the sun on the earth. The earth exerts the centrifugal force on the sun. Because of the enormous mass of the sun compared to that of the earth, this force produces little effect on the sun.

According to the modern constitution of the 'atom', it consists of a nuclear positive charge called 'proton', which is the seat of the mass of the atom. The atom is electrically neutral owing to a number of negative charges called 'electrons' of comparatively feeble mass revolving round the proton. The force of attraction exerted by the proton on the electron provides the centripetal force. The attraction of the latter on the former provides the centrifugal force.

The flattening of the earth at the poles and buldging at the equator is an effect of centrifugal force. The speed of the equatorial region is greater than that of the polar region, so the earth while in a plastic condition has buldged at the equator for greater centrifugal force and consequently flattened at the poles.

The well-known device **centrifuge** is used to separate substances of different densities held in suspension in a liquid. The vessel containing the liquid is rapidly rotated when substances of higher density than that of the liquid are driven away towards the sides of the vessel and those of lower density than that of the liquid are drawn towards the axis of rotation. In the familiar cream-separator, a vessel containing the milk is set to rotation. The cream which is of lower density than the liquid is drawn towards the axis of rotation and is easily removed.

The centrifugal drying machine consists of a cylindrical vessel with perforations on the sides. When moist cloths are placed into the vessel and the vessel is rapidly rotated, the water particles of the cloths are forced out through the perforations and they are quickly dried.

If a body moves in a curved path of any shape the centripetal force on the body changes from point to point on the path. At any point the centripetal force will still be given by mv^2/r where r is the radius of curvature of the path at that point and m and v have the usual significances.

Examples

1. A mass of one pound tied to one end of string 4 feet long, the other end of which is kept fixed, is whirled in a horizontal circle. If the breaking tension of the string is 128 lbs, wt., find the greatest speed which can be given to the weight.

Let v be the greatest speed which can be given in feet per sec. The centrifugal force exerted by the one ponnd mass when whirled with this speed

$$=\frac{1\times v^2}{4}\text{ poundals }\left(c.f.\ \frac{mv^2}{r}\right)$$

Since this is just the maximum force which the string can bear

$$\frac{1\times v^2}{4}=\underline{128\times 32}\quad (g=32\text{ ft/sec}^2)$$

$$\therefore\quad v^2=128\times 32\times 4\text{ or }(128)^2\ \therefore\ v=128\text{ ft. per. sec.}$$

2. A vertical over-bridge across a road has a radius of curvature 72 ft. at its top. Calculate the greatest speed in miles per hour at which a car can pass through the top of the bridge without leaving the ground. ($g=32$ ft/sec^2)

Let v be the greatest speed of the car at the top of the bridge in ft. per sec. The centrifugal acceleration of the car $=\frac{v^2}{72}$ ft/sec^2. The acceleration of gravity must balance it if the car remains just in contact with the bridge.

$$\therefore\quad \frac{v^2}{72}=32\qquad \therefore\quad v^2=72\times 32=(48)^2$$

$$v=48\text{ ft/sec}=\frac{48\times 60\times 60}{3\times 1760}\text{ or }32{\cdot}7\text{ miles per hour.}$$

3. Assuming that the moon describes a circular orbit of radius $3{\cdot}84\times 10^8$ metres in 27·3 days and the outer satelite of Mars describes a circular orbit of radius $2{\cdot}35\times 10^7$ metres in 1·26 days, find the ratio of the mass of Earth to that of Mars.

(Cambridge University)

Let M_e, M_m, M_M and M_s be the masses of the earth, moon, Mars and the satelite respectively in gms.

Since the gravitational attraction of the earth on the moon provides the centripetal force on the latter for its circular motion

$$G\frac{M_eM_m}{(3{\cdot}84\times10^8\times100)^2}$$

$$=M_m\times3{\cdot}84\times10^8\times100\left(\frac{2\pi}{27{\cdot}3\times24\times60\times60}\right)^2 \quad \ldots \quad (1)$$

(C. P. force$=mr\omega^2$ and G, the gravitational constant in C. G. S. units).

Also the gravitational attraction of the Mars on the satelite provides the centripetal force on the latter for its circular motion.

$$G\frac{M^{\mathrm{M}}M_s}{(2{\cdot}35\times10^7\times100)^2}$$

$$=M_s\times2{\cdot}35\times10^7\times100\left(\frac{2\pi}{1{\cdot}26\times24\times60\times60}\right)^2 \quad \ldots \quad (2)$$

Cancelling M_m in (1) and M_s in (2) from both sides and then dividing (1) by (2)

$$\frac{M_e}{M_{\mathrm{M}}}\,\frac{(2{\cdot}35)^2}{(3{\cdot}84)^2}\,\frac{1}{100}=\frac{3{\cdot}84\times(1{\cdot}26)^2\times10}{2{\cdot}35\times(27{\cdot}3)^2}$$

$$\therefore\quad\frac{M_e}{M_{\mathrm{M}}}=\frac{(3{\cdot}84)^3\times(1{\cdot}26)^2\times1000}{(2{\cdot}35)^3\times(27{\cdot}3)^2}$$

$\log(3{\cdot}84)^3=1{\cdot}7529$

$\log(1{\cdot}26)^2=\ {\cdot}2008$

$\log 1000\ =3{\cdot}0000$

$\log$ Num. $=4{\cdot}9537$

$\log(2{\cdot}35)^3=1{\cdot}1133$

$\log(27{\cdot}3)^2=2{\cdot}8724$

$\log$ Denom. $=3{\cdot}9857$

$$\log\frac{M_e}{M_{\mathrm{M}}}=4{\cdot}9537-3{\cdot}9857=0{\cdot}9680$$

$$\frac{M_e}{M_{\mathrm{M}}}=\text{antilog }0{\cdot}9680=9{\cdot}3\text{ nearly}$$

$\therefore$ mass of Earth $=9{\cdot}3\times$ mass of Mars.

Exercise

1. What are centripetal and centrifugal forces? Give some illustrations of them.

2. A string is 8 feet long and has one end attached to a fixed point on a smooth horizontal table. If a mass of 5 lbs. tied at the other end of the string describes uniformly a horizontal circle with a speed 5 feet per sec. Find the tension of the string. Ans. 15·6 poundals.

3. The tension of a string, one end of which is fixed and to the other end of which is attached a mass which revolves uniformly, is 9 times the weight of the revolving mass. Find the velocity of the mass if the length of the string be 2 feet. Ans. 24 ft per sec.

CHAPTER IV

SIMPLE HARMONIC MOTION

36. Periodic Motion—When a body moves in such a manner that is retraces its path after a regular interval of time, its motion is said to be *periodic*. The earth in course of its revolution round the sun retraces its path at a regular interval of time, *viz.*, one year. Thus, the motion of the earth round the sun is periodic. The motion of the piston of an engine, the motion of a pendulum, etc , are also periodic motions.

37. Oscillatory or Vibratory Motion—If the motion of a body is reversed in direction after a regular interval of time, it is called an *oscillatory* or *vibratory* motion. Thus, the oscillatory motion is a periodic '*to* and *fro*' (forward and backward) motion. The motion of a simple pendulum, the motion of the prong of a tuning-fork, etc., are reversed in direction after a regular interval of time. Hence, these are oscillatory motions. One complete '*to* and *fro*' motion is called an *oscillation* or *vibration*.

The term oscillatory motion is very often extended to denote any periodic motion. Thus, a periodic circular or elliptic motion is called a circular or elliptic vibration.

38. Simple Harmonic Motion—One type of vibratory motion which mostly interests us is the *simple harmonic motion* (S. H. M.). The motion is realised by the foot of a perpendicular drawn to a fixed straight line from a point describing a circle with uniform speed.

The term 'simple harmonic motion' is derived from the fact that the motion is realised by the orthogonal projection on a straight line of a point describing the simplest closed curve (circle) with the simplest speed (uniform speed) and from its connection with musical instruments.

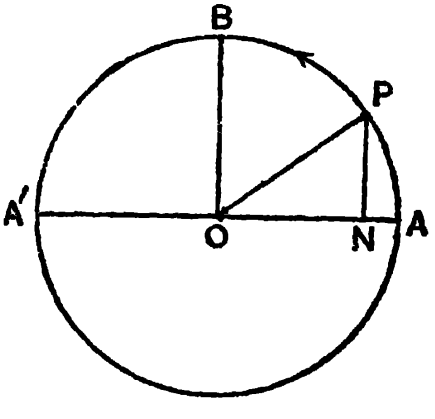

Fig. 9

Let a point P (fig. 9) describe the circle ABA' moving with uniform speed in the anti-clockwise direction. For convenience, take a diameter AOA' as the fixed straight line upon which the perpendicular PN from P is drawn. As P moves round and round the circumference of the circle, the foot N of the perpendicular PN executes a vibratory motion on AA', with O, the centre of the circle, as its mean position. The circle is called the **auxiliary circle** or the **circle of reference.**

39. **Amplitude**—It will be observed that N is shifted or displaced to different extents at different times from its mean position O. The maximum displacement or shift of the vibrating point from its mean position in either direction is called the *displacement amplitude* or simply *amplitude* of vibration. Thus, in figure 9, OA or OA' is the amplitude of vibration of N.

40. **Period**—The time taken by the vibrating point N (fig. 9) to execute one complete oscillation is called its *period.* It is evident that the period of oscillation of N is equal to the time which P takes to move once round the circle of reference.

41. **Frequency**—The number of oscillations (not necessarily a whole number) executed by the vibrating point (fig. 9) in unit time is called its *frequency*.

If T represents the period and n the frequency. the time taken to execute n oscillations is nT.

Again, these n oscillations are executed in unit of time.

$$\therefore \quad nT=1$$

$$\text{Hence,} \quad n=\frac{1}{T} \text{ or, } T=\frac{1}{n}$$

42. **Phase**—The *phase* of a vibrating point at any instant of time determines its *state of displacement and motion* at that instant of time. Thus, anything which will completely specify both the displacement of a vibrating point and its direction of motion at a certain time, will be a measure of its phase at that time.

In fig. 9 the phase of the vibrating point, when it is at N moving from right to left, is not identical with its phase when it is at the same position while moving from left to right ; because, although the displacement of the vibrating point is the same in the two conditions, still its states of motion are different—as the directions of motion in the two conditions are opposite to each other. Also the two conditions are not of opposite phases ; for, although the directions of motion are opposite still the displacements are not opposite. The vibrating point will have the same phase as in the former case again, when during the course of its vibration it will pass through N moving from right to left. This will evidently take place after the lapse of its periodic time or any multiple of the periodic time. Thus. the phase of a vibrating point is repeated after the lapse of its periodic time.

In fig. 10, P_1 and P_2 are the two positions of the point on the circle of reference such that P_1 and P_2 are diametrically opposite to each other. N_1 and N_2 are the projections of P_1 and P_2 on the diameter AOA'. When the point on the circle of reference passes through P_1 and P_2, the projections N_1 and N_2 have equal and opposite displacements and opposite directions of motion as shown by the arrow-heads near them. N_1 and N_2 are at opposite phases as they have equal and opposite displacements and opposite directions of motion. The time that elapses between the passages through P_1 and P_2 of the point on the circle of reference is half of the periodic time.. Hence, the phase of a vibrating point becomes opposite after half the periodic time.

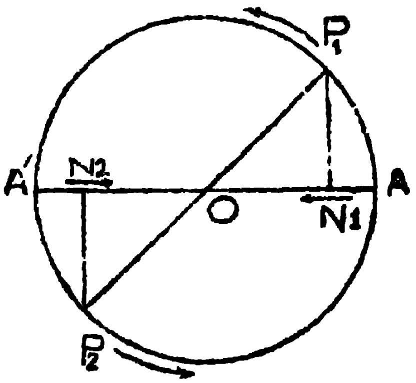

Fig. 10

Following a smilar construction as in fig. 10, it is easily observed that at the two extreme positions A and A', the vibrating point is at opposite phases. Also at the mean position O, the vibrating point when passing in the direction A to A' has a phase which is opposite to that when it passes in the direction A' to A.

It will be noted that the magnitude of velocity or amplitude is of no consideration for ascertaining the phase of a vibrating point. If two vibrating points pass through their *mean* positions at a particular time moving in the same direction, they are at that time at the same phase whatever the magnitudes of their velocities or amplitudes may be ; but if one of them moves in a direction opposite to that of the other, they are at opposite phases.

43. Measurement of phase—For measuring the phase we shall have to choose some standard state to reckon from, and also some method or methods which will bring about a quantitative distinction of the state in question from the standard state, with reference to the displacement and direction of motion of the vibrating point.

The standard state of the vibrating point is chosen* as that possessed by it when it has undergone maximum displacement in the positive direction. Thus, the standard state of N (fig. 9) is its position A, and in this position the point P on the auxiliary circle coincides with it. The measurement of phase commencing from its standard state is made in two ways :

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*See theory of Sound, Vol. 1 Rayleigh. Sometimes the standard state chosen is the mean position of the vibrating point moving in the positive direction (Text-book of Sound, Barton.)

(1) By the angle which the point on the auxiliary circle has subtended at the centre since it last passed through the standard position.

For, if we know this angle we can completely determine the position and direction of motion of the vibrating point. Thus, if the phase of the vibrating point at any instant of time is given as $\pi/3$, draw line OP inclined at $\pi/3$ (or 60°) to OA and from P on the auxiliary circle drop a normal PN on AA' ; N represents the position of the vibrating point. Further, if the motion of P on the circle is anti-clockwise, the motion of the vibrating point is evidently from right to left. Hence, the angle determines the phase of the vibrating point.

From what has been said in Art. 42, it will be observed that the phases $\frac{\pi}{2}$ and $\frac{5\pi}{2}$ are identical, as in these two cases, the vibrating point passes through the mean position O in the same direction, but phases $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ are opposite to each other as the vibrating point in these cases passes through the mean position O moving in opposite directions. *Thus, two phases become identical when they differ by* 2π *and opposite when they differ by* π.

(2) By the fraction of the periodic time which has elapsed since the point on the auxiliary circle last passed through the standard position.

For, if we know this time, we can also completely determine the position and direction of motion of the vibrating point. If the time elapsed since the point on the auxiliary circle last passed through the standard position A be $\frac{T}{4}$, the point on the auxiliary circle is at B where OB is perpendicular to AA' and the vibrating point is at O moving from right to left.

It is evident that phases $\frac{T}{4}$ and $\frac{5T}{4}$ are identical while phases $\frac{T}{4}$ and $\frac{3T}{4}$ are opposite to each other. Hence, phase is repeated after the periodic time but becomes opposite after one-half of the period.

44. Displacement—The displacement means the shift of the vibrating point from its mean position. In fig. 11, the displacement of N is ON. If a be the amplitude of the vibrating point, a is equal to the radius of the auxiliary circle. Also, if θ be the phase-angle AOP, we have representing the displacement by x,

$$x=a\cos\theta \qquad (1)$$

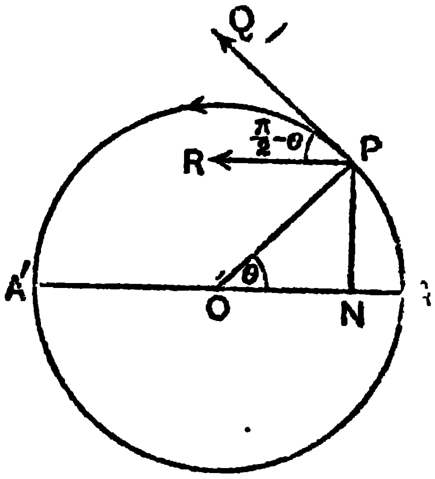

Fig. 11

Let ω be the angular velocity of P round O and let the time be counted from the instant P passes through the standard position A. If P travels the arc AP in time t, evidently $\theta=\omega t$. The displacement can, therefore, be put down as

$$x=a\cos\omega t \qquad \ldots \qquad \ldots \qquad (2)$$

Again, if T be the period of N, the point P moves once round the circle, and therefore, subtends an angle 2π at the centre in the time T. Hence, the angular velocity of P is given by $\frac{2\pi}{T}$ or, $\omega=2\pi n$ (where n is the frequency of N)

$$x=a\cos\frac{2\pi}{T}t \qquad (3)$$

$$x=a\cos 2\pi nt \qquad (4)$$

Equations (1), (2), (3) and (4) express the displacement in different forms.

45. Velocity—Since the motion of N (fig. 11) arises out of the motion of P, the velocity of N at any instant will be the resolved part of the velocity of P along AA' (the line of motion of N).

At the time t, the velocity of P is along the tangent PQ to the circle and its magnitude is $a\omega$. If PR is drawn parallel to AA', $\angle OPR=\theta$ and since $\angle OPQ$ is a right angle,

$$\angle QPR=\frac{\pi}{2}-\theta.$$

$\therefore$ The resolved part of the velocity of P along PR

$$=a\omega\cos\left(\frac{\pi}{2}-\theta\right)=a\omega\sin\theta=a\omega\sqrt{1-\cos^2\theta}$$

$$=a\omega\sqrt{1-\frac{x^2}{a^2}} = \omega\sqrt{a^2-x^2}=\frac{2\pi}{T}\sqrt{a^2-x^2}$$

This gives the magnitude of the velocity of N ; the direction of the velocity is along PR or AA'.

The maximum velocity of N corresponds to $x=0$ *i.e*, when it is passing through the mean position O. The maximum velocity or *velocity amplitude*$=\frac{2\pi a}{T}$.

Alternative Method—The velocity of N can be deduced easily by applying differential calculus. Since velocity is the rate of displacement with regard to time, velocity$=\frac{dx}{dt}$.

Again, $x=a\cos\omega t$ ... . (Relation 2, Art. 44)

∴ differentiating x with regard to t,

$$\text{velocity } V=\frac{dx}{dt}=-a\omega\sin\omega t \quad \text{... (1)}$$

$$=-a\omega\sqrt{1-\cos^2\omega t}$$

$$=-a\omega\sqrt{1-\frac{x^2}{a^2}}=-\omega\sqrt{a^2-x^2}=-\frac{2\pi}{T}\sqrt{a^2-x^2}\text{*}$$

46. Acceleration—The acceleration of N (fig. 11) will be given by the resolved part of the acceleration of P along the direction of motion of N. Following Art. 33, the acceleration of P is directed towards PO and is of magnitude $\frac{v^2}{a}$ or, $a\omega^2$, where v is the speed of P on the auxiliary circle. As the straight line PR is drawn parallel to AA', the acceleration $a\omega^2$ in the direction PO, if resolved in the direction PR, the resolved part will give the acceleration of N. Since $\angle OPR=\theta$, the resolved part along PR

$$=a\omega^2\cos\theta=\omega^2 x \quad \text{... (from 1, Art. 44)}$$

$$=\left(\frac{2\pi}{T}\right)^2 x$$

This gives the magnitude of acceleration of N. The direction of the acceleration of N is evidently along PR or AO, *i.e.*, along the negative direction of x.

---

[*] For a $+ve$ or $-ve$ value of x, velocity of N may be both along the $+ve$ or $-ve$ direction of x as the velocity depends upon $\sqrt{a^2-x^2}$ which may be either $+ve$ or $-ve$.

Alternative Method—The acceleration of N can be deduced easily by differentiating its velocity with regard to time as acceleration is defined as the rate of change of velocity with regard to time. The velocity of N at the time t is given by,

$$V = -a\omega \sin \omega t \quad \ldots \quad \ldots \quad \text{(Relation 1, Art. 45)}$$

$$\therefore \quad \text{acceleration} = \frac{dV}{dt} = -a\omega^2 \cos \omega t$$

$$= -w^2x = -\left(\frac{2\pi}{T}\right)^2 x \quad \ldots \text{(From Art. 44)}$$

This expression is independent of the motion of P on the circle of reference, an idea which we need no longer retain as an S. H. M. occurs independent of a circular motion.

The expression for acceleration shows that the acceleration is proportional to the displacement of the vibrating point from its mean position. The negative sign in the expression further indicates that the direction of acceleration is opposite to the direction of displacement, *i.e.*, the acceleration is always directed towards the mean position. From this has been derived the definition of *S. H. M.*

47. Definition of S. H. M.—The Simple Harmonic Motion is defined from the expression of acceleration. *If a body moves so that its acceleration is always directed to a fixed point in its path and is proportional to its displacement from the same fixed point, the motion of the body is called simple harmonic.*

From what we have seen in Art. 46, if the acceleration of the vibrating point is given by $-\mu x$,

$$\mu = \left(\frac{2\pi}{T}\right)^2$$

and the period of oscillation is given by

$$T = 2\pi\sqrt{\frac{1}{\mu}} = 2\pi\sqrt{\frac{x}{\mu x}} = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$

The frequency of oscillation will be given by $\frac{\sqrt{\mu}}{2\pi}$.

The expression for the period of oscillation shows that the period is independent of the amplitude of vibration. A vibratory motion of this type, in which the period of oscillation is independent of the amplitude is called an **isochronous motion**. The amplitude of vibration of a simple pendulum gradually decreases but each vibration is executed in the same time so long its amplitude of vibration is not large. Hence, the motion of a simple pendulum is isochronous.

48. Epoch—In the preceding articles, the phase-angle has been measured from the instant P and N passed through the standard position A. The time as well, was measured from the same instant. Often for generality there is introduced a difference between the beginnings of their measurements.

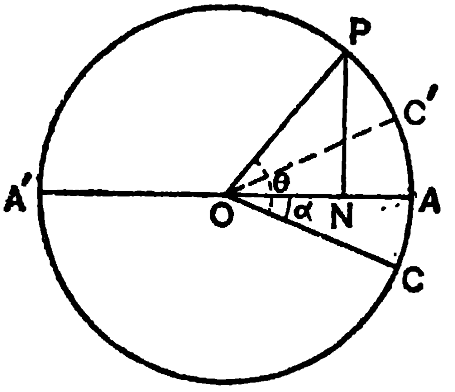

Fig 12

In fig. 12, let the measurement of the phase angle begin when P moves through a point C. Then the phase of the vibrating point at N which is denoted by θ is given by the angle COP. Let the measurement of the time commence when P passes through the standard position A. The angle COA which measures the phase at the commencement of time is called the *epoch*. Thus, *epoch is the phase at the commencement of time.*

Under this condition, the expression for displacement takes up the form

$$x = a \cos AOP = a \cos (COP - COA)$$
$$= a \cos (\theta - \alpha)$$

where α is the epoch of the $S.\ H.\ M.$

If, on the other hand, the phase angle is counted from the instant when P passes through C', $\angle C'OP$ is the phase θ and $\angle C'OA$ is the epoch α, and the displacement takes the form

$$x = a \cos POA = a \cos (\theta + \alpha).$$

49. We have so long considered the simple harmonic motion of a *point* in its path. Let us next direct our attention to the similar motion of a *body* having a definite *mass*. When such a massive body executes simple harmonic motion, the body is acted upon by a force which tends to bring it to the mean position of its motion and under this force the body is actually brought to its mean position in which it rests when the vibrations subside. Thus, a vibrating simple pendulum under the action of gravity comes to rest in its mean position when its vibrations die away. The prongs of a vibrating tuning-fork under elastic forces come to rest in their mean positions when the vibrations subside. A magnet vibrating in a uniform magnetic field ultimately rests in the mean position of its path under the action of the magnetic field. This mean position in which the body rests, on subsidence of its vibrations, is called its undisturbed position. We shall now calculate the force acting on the vibrating body when it is displaced from its undisturbed position.

50. Force—The expression for acceleration of the body is deduced in Art. 46 and is given by $-\omega^2 x$.

The force acting upon the body of mass m will, by the second law of motion, be given by

$$F = -m\omega^2 x = -\frac{4\pi^2 m}{T^2}\cdot x$$

The force is proportional to the displacement of the body ; the negative sign, as in the expression for acceleration, indicates that the force and the displacement are directed oppositely to each other.

51. Kinetic Energy—The kinetic energy of the body executing *S. H. M.*, at a certain position, will be given by $\frac{1}{2}mV^2$ where m is the mass of the body and V, its velocity in that position. Thus, when the displacement of the body is x, its kinetic energy will be given by

$$K.E. = \frac{1}{2}mV^2 = \frac{1}{2}m\omega^2(a^2 - x^2)\text{......(Art. 45)}$$

$$= \frac{2\pi^2 m}{T^2}(a^2 - x^2)$$

52. Potential Energy—The potential energy of the body at a certain position will be measured by the work done by the body in moving from that poition to its position of rest or undisturbed position in which it does not possess any potential energy. The calculation of the gravitational potential energy of a body is made by multiplying the weight of the body by its height from sea-level. There the force acting on the body. *viz.*, its weight remains constant throughout its path. But the force acting on a body executing simple harmonic motion depends upon its displacement and varies with its displacement proportionately. The work done in moving from the displaced position to the undisturbed position can, however, be measured by multiplying the mean force over its path by the length of the path. The force on the body in the displaced position is $m\omega^2 x$[*] and in the undisturbed position zero. Hence, the mean force over this path is $\frac{1}{2}m\omega^2 x$. The work, which the body can do in passing to its undisturbed position, which is a measure of its potential energy, will be given by

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[*] Since energy is a scalar quantity, the sign of the force is ignored while calculating the energy of the body.

$$P.E.=\frac{1}{2}m\omega^2 x\times x=\frac{1}{2}m\omega^2x^2=\frac{2\pi^2m}{T^2}x^2$$

Alternative Method—The potential energy can be otherwise calculated in the following way :

The force acting on the body when it is displaced through x is $m\omega^2x$. The work done by the body in moving a short distance, dx over which the force may be supposed to be constant, is $m\omega^2x.dx$. Therefore, the work done by the body in moving through x or its potential energy

$$=\int m\omega^2xdx=\frac{1}{2}m\omega^2x^2=\frac{2\pi^2m}{T^2}x$$

53. Total Energy—The total energy of the body is the sum of its kinetic and potential energies and is given by

$$T.E.=K.E.+P.E.=\tfrac{1}{2}m\omega^2(a^2-x^2)+\tfrac{1}{2}m\omega^2x^2$$
$$=\frac{1}{2}ma^2\omega^2=\frac{2\pi^2ma^2}{T^2}$$

The expression for total energy of the body is independent of x or the displacement of the body ; which shows that the energy of the body remains a constant throughout its path but it simply changes from kinetic to potential and *vice-versa*. This is a confirmation of the principle of conservation of energy.

54. Equation to S.H.M.—If x be the displacement of a moving body, at a time t, the velocity V of the body is given by $\frac{dx}{dt}$ and since the acceleration is the rate of change of velocity with regard to time ; the acceleration can be put down as $\frac{dV}{dt}$ or, $\frac{d}{dt}\left(\frac{dx}{dt}\right)$ or, $\frac{d^2x}{dt^2}$.

From Art. 46.

$$\frac{d^2x}{dt^2}=-\omega^2x \text{ or, } \frac{d^2x}{dt^2}+\omega^2x=0 \qquad \dots \quad \dots \qquad (1)$$

This is known as the differential equation of S. H. M.

To solve this equation, put μ for ω^2 and multiply both sides by $2\frac{dx}{dt}$

$$2\frac{d^2x}{dt^2}\cdot\frac{dx}{dt}+2\mu x\frac{dx}{dt}=0$$

Integrating,

$$\left(\frac{dx}{dt}\right)^2+\mu x^2=c\ldots\ldots\ldots\ldots(2)$$ (where c is an integration constant).

To evaluate c, we see that as the acceleration and displacement are oppositely directed, the velocity will vanish after some displacement ; let the velocity $\left(\frac{dx}{dt}\right)$ of the body be zero when $x=a$; a is known as the amplitude.

Putting these values in equation (2)

$$0+\mu a^2=c \qquad \text{or,} \quad c=\mu a^2$$

Restoring the value of c in equation (2) and re-arranging the terms,

$$\left(\frac{dx}{dt}\right)^2=\mu(a^2-x^2)$$

$$\text{Or,}\ \frac{dx}{dt}=\sqrt{\mu}\sqrt{a^2-x^2}\ \text{Or,}\ \frac{dx}{\sqrt{a^2-x^2}}=\sqrt{\mu}dt$$

Integrating both sides,

$$\sin^{-1}\frac{x}{a}=\sqrt{\mu}t+\delta$$ (where δ is an integration constant)

$$\text{Or,}\ \frac{x}{a}=\sin(\sqrt{\mu}t+\delta)$$

$$\text{Or,}\ x=a\sin(\sqrt{\mu}t+\delta)$$

$$=a\sin\sqrt{\mu}t\cos\delta+a\cos\sqrt{\mu}t\sin\delta$$

$$=A\sin\sqrt{\mu}t+B\cos\sqrt{\mu}t$$ (where A and B are constants)

$$=A\sin\omega t+B\cos\omega t,$$ where ω is written for $\sqrt{\mu}$,

This is the general solution of equation (1) and represents the general equation of a simple harmonic motion.

The particular solutions are

(1) $x=A\sin\omega t$ and (2) $x=B\cos\omega t$

any of which represents a simple harmonic vibration.

Period—From the general equation to S. H. M.

$x=A\sin\omega t+B\cos\omega t$

$x=A\sin(\omega t+2\pi)+B\cos(\omega t+2\pi)$

$$=A\sin\omega\left(t+\frac{2\pi}{\omega}\right)+B\cos\omega\left(t+\frac{2\pi}{\omega}\right)$$

Hence, the displacement x is repeated in magnitude and direction after an interval $\frac{2\pi}{\omega}$ which is therefore the period T of the *S.H.M.* (Art. 43.)

$\therefore\quad T=\frac{2\pi}{\omega}$ or' $\frac{2\pi}{\sqrt{\mu}}$.......... Art. 47.

The frequency n of the *S. H. M.* is given by

$n=\frac{\omega}{2\pi}$ or, $\frac{\sqrt{\mu}}{2\pi}$.........Art, 47.

Amplitude—If the equation to an S.H.M. is given in the form $x=A\sin\omega t$ or $x=B\cos\omega t$, A and B represent amplitudes in in the two respective cases as these are the maximum values of x for all times. If the equation is given as $x=A\sin\omega t+B\cos\omega t$, the amplitude is $\sqrt{A^2+B^2}$ (See example 2 below).

Examples

1. A point describes simple harmonic vibrations on a line 4 cm. long. Its velocity when passing through the centre of the line is 12 cm. per second. Find the period, (C. U. 1919)

Amplitude a of the *S. H. M.*=2 cm.

The velocity of the point at a displacement $x=\frac{2\pi}{T}\sqrt{a^2-x^2}$

Hence velocity when passing through the centre of the line $=\frac{2\pi}{T}a$ (for $x=0$)

$\therefore\quad 12=\frac{2\pi}{T}\times 2$ or $T=\frac{\pi}{3}$ sec=1·042 sec.

★2. If the displacement of a moving point at any time is given by the equation $x=a\cos\omega t+b\sin\omega t$.

Show that the motion is simple harmonic; If $a=3$, $b=4$ and $\omega=2$, determine the period, amplitude, max. velocity and max. acceleration of the point. (Madras 1949)

The above equation is the general equation to *S. H. M.* (see Art 54).

Let the equation be represented by $x = A\cos(\omega t - \alpha)$ where A represents the amplitude and α, the epoch (Art. 48).

$\therefore \quad A\cos(\omega t - \alpha) = a\cos\omega t + b\sin\omega t$

or, $\quad A\cos\omega t\cos\alpha + A\sin\omega t\sin\alpha = a\cos\omega t + b\sin\omega t$

Equating coefs. of $\cos\omega t$ and $\sin\omega t$ respectively

$A\cos\alpha = a$ and $A\sin\alpha = b$

Squaring and adding $\quad A^2 = a^2 + b^2$

$\therefore \quad A = \sqrt{a^2+b^2} = \sqrt{3^2+4^2} = 5$

$$\text{Period} = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi = 3{\cdot}142.$$

Max. velocity (from Art '45) $= A\omega = 5 \times 2 = 10$

Max acceleration (from Art '46) $= \omega^2 A = 20$.

3. A particle is moving with simple harmonic motion in a straight line. When the distance of the particle from the equilibrium position has the values x_1 and x_2 the corresponding values of the velocity are u_1 and u_2. Show that the period is

$$2\pi\,[(x^2{}_2 - x_1{}^2)/(u_1{}^2 - u^2{}_2)]^{\frac{1}{2}}$$

Find also (i) the maximum velocity and (ii) the amplitude.

(*Madras 1959*)

Let a be the amplitude and T, the period of the motion. From the expression for velocity (Art. 45)

$$u_1 = \frac{2\pi}{T}\sqrt{a^2 - x_1{}^2} \quad \text{or,} \quad u_1{}^2 = \frac{4\pi^2}{T^2}(a^2 - x_1{}^2) \qquad \ldots (1)$$

$$\text{Also} \quad u_2 = \frac{2\pi}{T}\sqrt{a^2 - x_2{}^2} \quad \text{or,} \quad u_2{}^2 = \frac{4\pi^2}{T^2}(a^2 - x_2{}^2) \ldots (2)$$

$$\text{Subtracting} \quad u^2{}_1 - u_2{}^2 = \frac{4\pi^2}{T^2}(x_2{}^2 - x_1{}^2)$$

$$\text{or,} \quad \frac{T^2}{4\pi^2} = (x_2{}^2 - x_1{}^2)/(u_1{}^2 - u_2{}^2)$$

$$\text{or,} \quad T = 2\pi\,[(x_2{}^2 - x_1{}^2)/(u_1{}^2 - u_2{}^2)]^{\frac{1}{2}}$$

Substituting for T in (1)

$$u_1{}^2 = (a^2 - x_1{}^2)\,\frac{u_1{}^2 - u_2{}^2}{x_2{}^2 - x_1{}^2}$$

$$\therefore\ a^2\frac{u_1^2-u_2^2}{x_2^2-x_1^2}=u_1^2+\frac{x_1^2(u_1^2-u_2^2)}{x_2^2-x_1^2}=\frac{x_2^2u_1^2-x_1^2u_2^2}{x_2^2-x_1^2}$$

$$\therefore\ \text{amp. } a=[(x_2^2u_1^2-x_1^2u_2^2)/(u_1^2-u_2^2)]^{\frac{1}{2}}$$

$$\text{Max. vel. (from Art. 45)}=\frac{2\pi}{T}a=\left(\frac{u_1^2-u_2^2}{x_2^2-x_1^2}\right)^{\frac{1}{2}}\left(\frac{x_2^2u_1^2-x_1^2u_2^2}{u_1^2-u_2^2}\right)^{\frac{1}{2}}$$

$$=[(x_2u_1^2-x_1u_2^2)/(x_2^2-x_1^2)]^{\frac{1}{2}}.$$

4. A particle executes *S. H. M.* of period 16 secs. Two seconds after it passes the centre of oscillation, its velocity is found to be 4 ft. per sec. find the amplitude. (*Madras 1949*)

The period being 16 secs., its time for moving from extreme position to centre of oscillation is 4 secs. So at the time 2 secs. after it passes the centre of oscillation, it is at 2 secs. interval from the nearest extreme position,

In the expression for displacement $x=a\cos\omega t$, $t=2$ secs.

The velocity $\frac{dx}{dt}=-a\omega\sin\omega t$

Ignoring the $-ve$ sign and putting

$$\frac{dx}{dt}=4 \text{ ft/sec, } \omega=\frac{2\pi}{T},\ T=16 \text{ sec.}$$

$$4=a\frac{2\pi}{16}\sin\frac{2\times2\pi}{16}\quad\text{or}\quad 4=\frac{\pi a}{8}\sin\frac{\pi}{4}$$

$$\therefore\ a\text{ (amplitude)}=\frac{32\sqrt{2}}{\pi}=14{\cdot}4\text{ ft. nearly.}$$

Exercise

1. Define the following :—Period, amplitude, phase. Show that the motion of the bob of a simple pendulum is approximately simple harmonic.
[*C. U. 1915, 21, 23*]

A particle moving with simple harmonic motion has a period of 0·001 sec. and amplitude 0·5 cm. Find its acceleration when it is 0·2 cm. from its mean position and its maximum velocity. *Ans.* 79×10^5 cms/sec^2 ; 3141 cms/sec.
[*C. U. 1915*]

2. Define amplitude; phase and period of a simple harmonic motion. Find the period, given that the magnitude of the force at a distance x is μx.

A pendulum which beats once in 2 seconds at A gains two beats an hour at B ; compare the weights of the same substance at the two places
***Ans.* 9 : 9·02. [*C. U. 1916*]**

3. Find the velocity and acceleration of a point executing simple harmonic motion. Show that the motion of a simple pendulum oscillating with small amplitude is simple harmonic in character.

A point describes simple harmonic vibrations in a line 4 cm. long. Its velocity when passing through the centre of the line is 12 cm. per second. Find the period. *Ans.* $\pi/3$ sec. [*C. U. 1919*]

4. Distinguish between Oscillatory Motion and Periodic Motion.

Find an expression for the displacement at any instant of a particle executing Simple Harmonic Motion of known period and amplitude. Explain how the time-displacement curve can be graphically drawn. [*C. U. 1922*]

5. Show how to represent graphically the changes in the potential and kinetic energies of a particle executing simple harmonic motion.

A particle is suspended from a point by means of an inextensible string of length 120 cm. It is projected horizontally with a velocity of 30 cm. per second. Find the height to which it will rise. *Ans.* 0·46 cm. [*C. U. 1925*]

6. What do you mean by simple harmonic motion? What is the nature of the force that will make a particle move in simple harmonic motion? What reasons are there for believing that the motions that give rise to sound and light are simple harmonic? [*C. U. 1926*]

7. A particle P moves with a constant speed on a circle of radius 10 cm., making 8 complete revolutions per second. Discuss the motion of the projection Q of the particle P on a diameter of the circle, and find an expression for the velocity of Q at any instant. Calculate the position and the velocity of Q at the end of the first, second and third seconds. [*C. U. 1927*]

8. Deduce the equation for the simple harmonic motion of a particle. [*C. U. 1940*]

9. Define simple harmonic motion and explain what is meant by the period, amplitude and phase of such a motion? (C. U. 1952,59)

If the period of S. H. M is 12 seconds and amplitude 10 cm. what are the phase and displacement at a time 14 secs. after a passage of the particle through its extreme positive elongation? *Ans.* 60°, 5 cm. [*C. U. 1952*]

CHAPTER V

COMPOSITION OF SIMPLE HARMONIC OSCILLATIONS

55. Composition of two Collinear Simple Harmonic Motions of Equal Periods differing in Amplitude and Phase.

Suppose a particle is acted upon simultaneously by two forces, which, if acted separately, would cause the particle to execute two separate S. H. M.'s along the same line. Let us study the resulting motion of the particle under the two forces acting together.

The solution of the problem can be obtained in two ways,—(*a*) by graphical construction and (*b*) by analytical method.

(*a*) *Graphical construction*—

In fig. 13, the two simple harmonic motions are executed on the straight line XOX'. The auxiliary circle of one *S. H.. M.*, is $A_1P_1A'_1$ and of the other is $A_2P_2A'_2$. The amplitudes of the two *S.H.M.*'s are OA_1 and OA_2 respectively. P_1 and P_2 are the positions at a certain instant of time, of the two points revolving anti-clockwise on their auxiliary circles, each with the same period

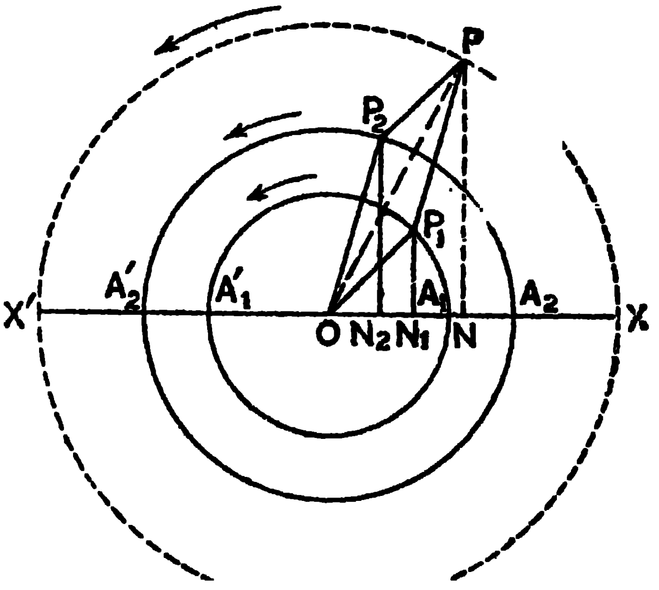

Fig 13

as that of the either *S.H.M.* The phases of the two *S.H.M.*'s at that time are $\angle P_1OA_1$ and $\angle P_2OA_2$ respectively, so that $\angle P_1OP_2$ measures the difference of phase between the two *S.H.M.*'s. The instantaneous displacements of the two *S.H.M.*'s are given by ON_1 and ON_2.

Since the two component displacements are along the same straight line, the resultant displacement by the principle of superposition,* will be equal to their sum. To obtain the resultant displacement, complete the parallelogram OP_1PP_2 and from P drop a perpendicular PN on XX'. Since OP_2 and P_1P are equal and parallel, their projections ON_2 and N_1N on XX' are equal.

$$\therefore \quad ON = ON_1 + N_1N = ON_1 + ON_2$$

ON gives the resultant instantaneous displacement.

Thus, if we suppose the point P to rotate uniformly in a circle in the anti-clockwise direction with O as centre, in a period which is equal to the period of either P_1 or P_2, the projection of OP on XX' will give the resultant displacement at any time. The resultant motion is thus simple harmonic having the same period as that of either of the component *S.H.M.*'s. OP measures the amplitude of the resultant motion and $\angle POX$ the instantaneous phase.

In the triangle P_1OP,

$$OP^2 = OP_1^2 + P_1P^2 - 2OP_1.P_1P \cos OP_1P$$

$$\text{Or,} \quad OP^2 = OP_1^2 + OP_2^2 + 2OP_1.\ OP_2 \cos P_1OP_2.$$

This gives OP, the amplitude of the resultant motion, in terms of the amplitudes and phase-difference between the component motions. As P_1, P_2 and P move on their respective circles with the same period, the angle POP_1 is a constant and is the difference in phase between the resultant *S.H.M.* and that of N_1. If a perpendicular PQ (not shown) is drawn from P on OP_1 (produced).

$$\tan POP_1 = \frac{PQ}{OQ} = \frac{PP_1 \sin PP_1Q}{OP_1 + PP_1 \cos PP_1Q}$$

$$= \frac{OP_2 \sin P_1OP_2}{OP_1 + OP_2 \cos P_1OP_2}$$

This gives the angle POP_1, which measures the difference in phase between the resultant S.H.M. and that of N_1.

(*b*) *Analytical method*—

Let the amplitudes of the two S.H.M.'s along the same straight line be a and b respectively. At a given instant of time let θ represent the phase of the former and $(\theta+\delta)$ that of the latter, so that the difference of phase between the two motions is δ. If x_1 and x_2 respectively represent the displacements of the two at that instant of time, we have

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*See Author's Text-Book of Sound, Art. 26.

$$x_1=a\cos\theta \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\text{and } x_2=b\cos(\theta+\delta) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Since the displacements are along the same straight line, the resultant displacement $x=x_1+x_2$

$$\begin{aligned} \therefore\quad x &= a\cos\theta+b\cos(\theta+\delta) \\ &= a\cos\theta+b\cos\theta\cos\delta-b\sin\theta\sin\delta \\ &= (a+b\cos\delta)\cos\theta-b\sin\theta\sin\delta \end{aligned}$$

Let us choose c and γ such that

$$a+b\cos\delta=c\cos\gamma \quad \ldots \quad \ldots \quad (3)$$

$$\text{and } b\sin\delta=c\sin\gamma \quad \ldots \quad \ldots \quad (4)$$

$$\begin{aligned} \therefore\quad x &= c\cos\theta\cos\gamma-c\sin\theta\sin\gamma \\ &= c\cos(\theta+\gamma) \quad \ldots \quad \ldots \quad \ldots \quad (5) \end{aligned}$$

The resultant motion is thus simple harmonic, having c for its amplitude and $\theta+\gamma$ for its instantaneous phase or γ the difference of phase between the resultant motion and the motion represented by equation (1).

To evaluate c, square (3) and (4) and add together, when

$$c^2=a^2+b^2+2ab\cos\delta \quad \ldots \quad \ldots \quad (6)$$

To evaluate γ, divide (4) by (3), when

$$\tan\gamma=\frac{b\sin\delta}{a+b\cos\delta} \quad \ldots \quad \ldots \quad (7)$$

Equations (6) and (7) give the values of c and γ. The values are analogous to those obtained by the graphical method.

Special Cases—The following special cases are of interest :

(1) If $\delta=0$, $c=a+b$; *i.e.*, the resultant amplitude is the sum of the component amplitudes.

(2) If $\delta=\pi$, $c=a\sim b$; *i. e.*, the resultant amplitude is the difference of the component amplitudes.

(3) If $\delta=\pi$ and $a=b$, $c=0$; *i.e.*, the component S.H.M.'s cancel each other.

(4) If $\delta=\frac{\pi}{2}$, $c^2=a^2+b^2$.

56. It the periods of component S.H.M.'s represented by (1) and (2) of Art. 55, b are not equal but as $n:m$, it is evident that when P_1 moves through an angle $m\theta$ on its auxiliary circle, P_2 moves through an angle $n\theta$ on its circle of reference. The equations (1) and (2) take the forms,

$$x_1=a\cos m\theta$$

$$\text{and } x_2=b\cos(n\theta+\delta)$$

The resultant displacement is given by

$$x = x_1 + x_2 = a \cos m\theta + b \cos (n\theta + \delta).$$

The expression is *periodic* if the periods be *commensurable*. But it does not, in general, admit of any useful reduction. It represents what is termed a *compound harmonic* motion[*].

47. Composition of two Rectangular S. H. M's of Equal Periods differing in Amplitude and Phase.

Suppose a particle is acted upon simultaneously by two forces which, if acted separately, would cause the particle to execute two separate *S. H. M.*'s in perpendicular directions. Let us study the resulting motion of the particle under the two forces acting together.

The solution of the problem as before, can be obtained in two ways,—(*a*) by graphical construction and (*b*) by analytical method.

(*a*) *Graphical construction*—

The graphical method of solving the problem is illustrated in fig. 14. Two straight lines *AB* and *BC* are drawn perpendicular

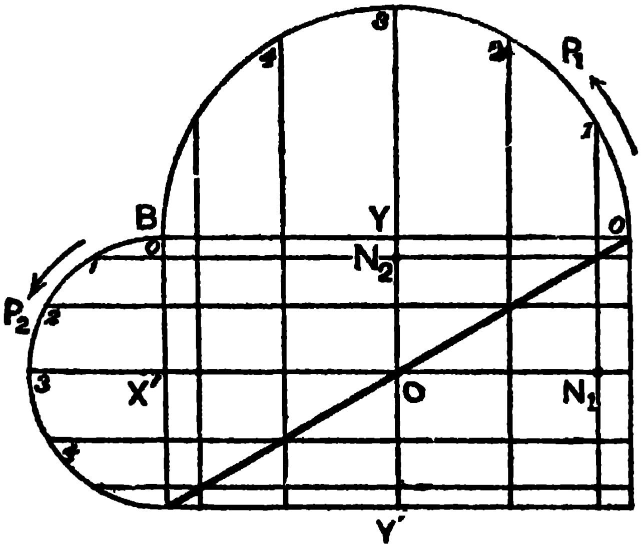

Fig. 14

to each other, of lengths equal to twice the amplitudes of the *S. H. M.*'s executed along *X* and *Y*-axes respectively. The rectangle *ABCL* is completed. With *AB* and *BC* as diameters, semi-circles AP_1B and BP_2C are drawn respectively. P_1 and P_2 are the tracing points moving anti-clockwise on these semi-circles. Since the periods of the two *S. H. M.*'s are the same, the points P_1 and P_2 travel over their respective semi-circles or the same fraction of their semi-circles in the same time. Thus, if each of the semi-circles be divided into the same number of parts, the points P_1 and P_2 will describe their respective parts in the same time.

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[*] See Author's Text-Book of Sound Art. 127.

In fig. 14, each of the semi-circles has been divided into six equal parts and through each division on the semi-circle AP_1B, a straight line parallel to BC is drawn to meet the line CL. Similarly, through each division on the semi-circle BP_2C, a straight line parallel to AB is drawn to meet the line AL. The rectangle $ABCL$ is thus divided into a large number of small rectangles.

The two *S. H. M.*'s are illustrated by the motion of the projection N_1 of P_1 moving on the central line XOX' and by the motion of the projection N_2 of P_2 moving on the central line YOY'. As P_1 is moving on its semi-circle from A to B its projection N_1 executes half its vibration from X to X', the other half of N_1's vibration is a return from X' to X, which may be imagined by supposing P_1 to move from B to A along the semi-circle. Similarly, as P_2 moves from B to C along the semi-circle, its projection N_2 on YOY' moves from Y to Y'. The other half of N_2's vibration is a return from Y' to Y, when P_2 moves from C to B back along the semi-circle.

When there is no phase-difference between the two *S H.M.*'s the points N_1 and N_2 begin their motions simultaneously from their positive extreme positions X and Y respectively. The points P_1 and P_2 at this stage are occupying the positions at A and B respectively. The successive positions of P_1 and P_2 are marked in the diagram by numbers 0, 1, 2. etc.

At the start the displacements due to the two component *S. H. M.*'s are respectively OX and OY and the resultant displacement, by the parallelogram law, is given by OA. Hence, the position of the point under the two *S H.M.*'s is A. When P_1 and P_2 are at the positions marked 1 on their respective semi-circles, the component displacements are ON_1 and ON_2 as shown in the the figure. The position of the point under the two *S.H.M.*'s is evidently the corner opposite to A of the small rectangle on which A is situated and so on for other successive positions of the point under the two *S. H. M.*'s. Joining these positions the resulting motion is given by the thick line AC.

Fig. 15 is drawn when there is a phase-difference $\frac{\pi}{6}$ between the two *S.H.M.*'s. The construction is exactly similar as in fig. 14, the only difference is that the phase of one of the *S.H.M.*'s, say of N_2, is $\frac{\pi}{6}$ in advance of that of N_1. This phase-difference is introduced by supposing that when P_2 has already moved to its position marked 1, the motion of P_1 begins from the position marked 0, because the angle subtended by each part of the semi-circle at the centre is $\pi/6$. The component displacements are ON_1 and ON_2. The resultant displacement is ON. Commencing from N and joining the opposite corners of the small rectangles, the resultant motion is represented by the elliptic curve.

If the phase-difference amounts to $\frac{\pi}{4}$, divide the semi-circles into four or eight equal parts and introduce the given phase-difference as before.

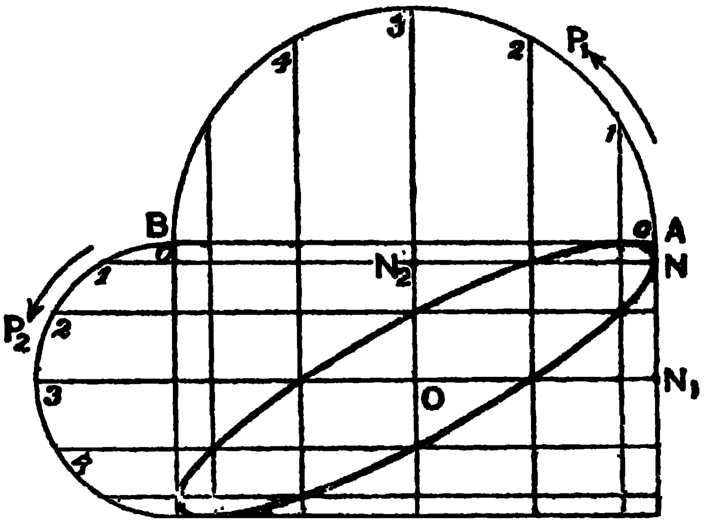


Fig. 15

(b) *Analytical Method*—

Let the two S. H. M.'s be executed along the X and Y-axes and their amplitudes respectively be a and b. At a particular instant of time, let θ be the phase of the S.H.M. along X-axis and $(\theta+\delta)$ be that of the other along the Y-axis. The difference of phases between the two S.H.M.'s is thus δ. Then, if x and y represent their respective displacements, the displacement equations of the two S.H.M.'s can be put down as

$$x = a \cos \theta \qquad \ldots \qquad \ldots \qquad \ldots \quad (1)$$

$$\text{and } y = b \cos (\theta + \delta) \qquad \ldots \qquad (2)$$

From (2), $\frac{y}{b} = \cos \theta \cos \delta - \sin \theta \sin \delta$

$$= \frac{x}{a} \cos \delta - \sqrt{1 - \frac{x^2}{a^2}} \sin \delta \qquad \ldots \quad (\text{from } 1)$$

Transposing,

$$\frac{y}{b}-\frac{x}{a}\cos\delta=-\sqrt{1-\frac{x^2}{a^2}}\sin\delta$$

Squaring and transposing again,

$$\frac{y^2}{b^2}-2\frac{xy}{ab}\cos\delta+\frac{x^2}{a^2}(\cos^2\delta+\sin^2\delta)=\sin^2\delta$$

$$\text{or,}\quad \frac{x^2}{a^2}-\frac{2xy}{ab}\cos\delta+\frac{y^2}{b^2}=\sin^2\delta \quad \ldots \quad \ldots \quad (3)$$

Relation (3) is the general equation to an ellipse. The resulting motion is thus elliptic in general.

Special Cases—The following special cases are of interest.

(1) If $\delta=0$, $\sin\delta=0$ and $\cos\delta=1$.

Putting these values, equation (3) reduces to

$$\frac{x^2}{a^2}-\frac{2xy}{ab}+\frac{y^2}{b^2}=0 \quad \text{or.} \quad \left(\frac{x}{a}-\frac{y}{b}\right)^2=0$$

$$\therefore \quad y=\frac{b}{a}x.$$

This represents a straight line passing through the origin and inclined at an angle $\tan^{-1}\frac{b}{a}$ to the axis of X (See fig. 14).

(2) If $\delta=\pi$,

equation (3) reduces to $y=-\frac{b}{a}x$.

This is also a straight line passing through the origin but inclined at an angle $\tan^{-1}\left(-\frac{b}{a}\right)$ to the axis of X.

(3) If $\delta=\frac{\pi}{2}$,

equation (3) reduces to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

This represents an ellipse of which the axes of symmetry are coincident with the co-ordinate axes and of which the semi-axes are equal to a and b.

(4) If $\delta=^{\pi}$ and $b=a$.

equation (3) reduces to $x^2+y^2=a^2$

which represents a circular motion. Conversely, a uniform circular motion can be resolved into two rectangular *S. H. M.*'s of the same amplitude and period but differing in phase by $\pi/2$. (See Art. 59)

58. Composition of two Rectangular S. H. M.'s differing in Amplitude, Phase and Period.

(*a*) *Graphical Construction*—

If the periods of the component *S. H. M.*'s are not the same, the graphical construction for the resultant motion can be made as in Art. 57(*a*). The only difference is that the semi-circles AP_1B and BP_2C will have to be divided into number of divisions having the same ratio as that of the periods of the component *S. H. M.*'s. For example, if the periods of the component *S H. M.*'s along the *X* and *Y*-axes be as 2 : 1, the convenient number of divisions of the semi-circles AP_1B and BP_2C will be 8 and 4 respectively (fig. 16). Thus, when P_1 moves through the 8 divisions from *A* to *B*, P_2 moves through 8 divisions from *B* to *C* and from *C* to *B* back on the semi-circle. Hence, in the time N_1 makes half of its oscillation, N_2 executes one complete oscillation ; or the period N_1 is twice that of N_2.

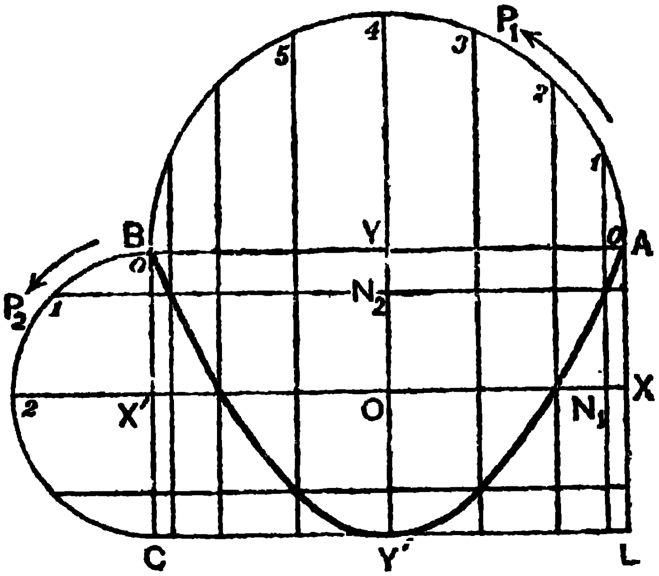


Fig. 16

It will, however, be noted that any phase-difference introduced between the two S. *H. M.*'s does not remain constant, because owing to the difference in periods of the two component motions, the phase-difference will be continually changing. We can, however, introduce a phase-difference at the start and study the resultant motion.

Fig. 16 is drawn when the periods of the two *S. H. M.*'s are as $2 : 1$ and for no phase-difference at the start. Proceeding as in Art. 57 the resultant motion is found to be parabolic.

Fig. 17 is drawn when the periods are $2 : 1$ and for an initial phase-difference of $\frac{\pi}{3}$, the phase of the *S. H. M.* along *Y*-axis being in advance. Here the numbers of divisions of the respective semi-circles are 6 and 3.

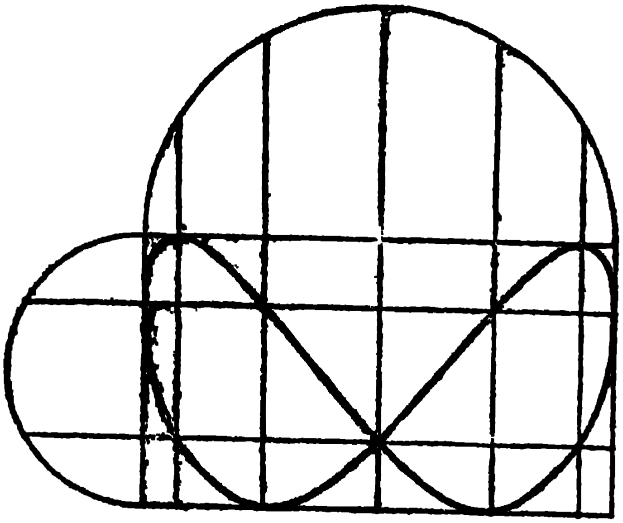


Fig. 17

Fig. 18 is drawn when the periods are $3 : 2$ and for no initial phase-difference. The numbers of divisions of the semi-circles are 6 and 4 respectively.

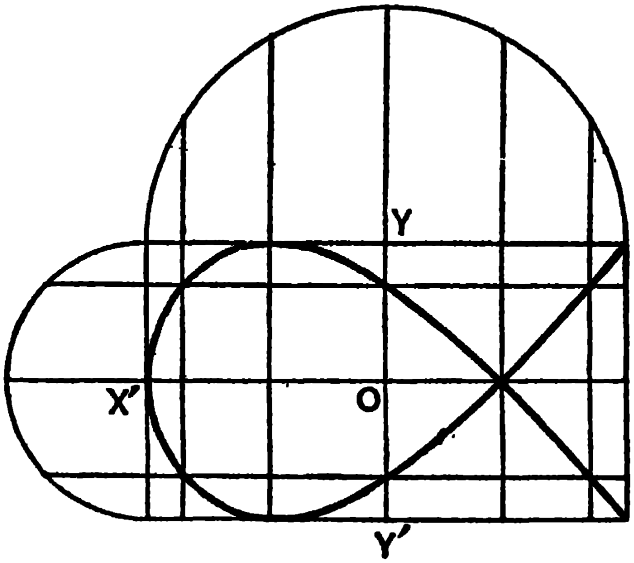


Fig. 18

(*b*) *Analytical Method*—

If the periods of the two *S. H. M.*'s executed along X and Y-axes are in the ratio 2 : 1, it is evident that when P_1 moves through an angle θ on its auxiliary circle, P_2 moves through an angle 2θ on its circle of reference. Thus, the instantaneous phases of the two *S. H. M.*'s are θ and 2θ respectively. The displacement equations for no initial phase-difference can be put down as,—

$$x=a\cos\theta \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

$$\text{and } y=b\cos 2\theta \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2)$$

$$\therefore \quad y=b(2\cos^2\theta-1)=b\left(2\frac{x^2}{a^2}-1\right) \quad \text{from (1)}$$

$$\text{or, } y=\frac{2b}{a^2}x^2-b \qquad \ldots \qquad \ldots \qquad \ldots \qquad (3)$$

This gives the resultant motion. It will be observed that the curve representing the resultant motion is a parabola symmetrical about the axis of Y having the point $(0,-b)$ as its vertex. This is found in fig. 16.

If the periods are 2 : 1 and if there is an initial phase-difference of δ, the equations (1) and (2) take the forms

$$x=a\cos\theta$$

$$\text{and } y=b\cos(2\theta+\delta)$$ (the phase of the *S. H. M.* along Y-axis being in advance).

Eliminating θ between the two equations

$$\frac{y^2}{b^2}-2\frac{y}{b}\left(2\frac{x^2}{a^2}-1\right)\cos\delta+\left(2\frac{x^2}{a^2}-1\right)^2=\sin^2\delta$$

If $\delta=0$, this equation reduces to

$$\left\{\frac{y}{b}-\left(2\frac{x^2}{a^2}-1\right)\right\}^2=0$$

which represents two coincident parabolas, the equation to the either parabola is

$$\frac{y}{b}=\frac{2x^2}{a^2}-1$$

$$\text{or, } y=\frac{2b}{a^2}x^2-b \ldots \text{(similar to equation 3).}$$

If the periods are 3 : 2, the displacement equations of the component motions are

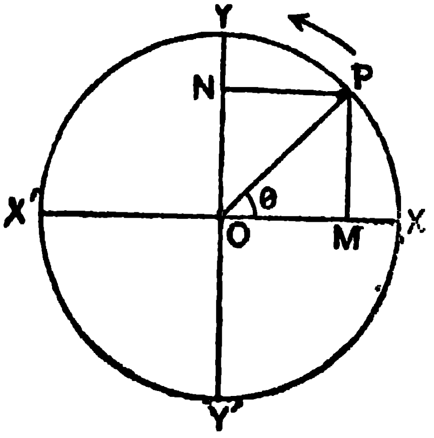
$$x=a\cos 2\theta$$

$$\text{and } y=b\cos 3\theta$$

Eliminating θ between the two equations, it will be found that the resultant curve is a cubic, represented by the equation

$$\frac{y^2}{b^2}=\frac{2x^3}{a^3}-\frac{3x}{2a}+\frac{1}{2}$$

59. Resolution of a Uniform Circular Motion—Let a point move uniformly in a circle and starting from X let it occupy a position P (fig. 19) at an instant of time t. Draw PM and PN perpendiculars to the rectangular diameters XOX' and YOY' of the circle. Then by the law of vector combination, the vector OP is equivalent to the vectors OM and ON. As P is moving round the circle, M executes an *S.H.M.* along the line XOX' and N executes an *S. H.M.* along the line YOY'. At any time the vector representing the distance of P from O will be equivalent to the vectors representing the displacements of M and N from the same point O. Hence, the circular motion of P is equivalent to the *S.H.M.*'s of M and N along XOX' and YOY' respectively.



Fig, 19

It will be observed that when P is at X, M is at X, its phase is zero. At this time N is at O passing from Y' to Y; its phase is the reflex angle $YX'Y'X$ or $3\pi/2$. Hence, the phase of N is $3\pi/2$ in advance of that of M. Since phase is unaltered by a change of 2π (Art. 43.) the phase of N may be regarded as $3\pi/2-2\pi$ or $-\pi/2$ *ahead of* or $\pi/2$ *behind* that of M.

The above conclusion may also be arrived at more precisely in the following way :—If θ be the phase of M at the time t, then θ is the angle POX (Art. 43). The displacement of M along x-direction is given by

$$x=a\cos\theta \text{ (where } a\text{=amplitude of } M).$$

At the same instant of time t, the point N is moving along the direction OY. Since the phase measurement of N will commence from the instant P passes through Y (Art. 43), the phase of N is given by the reflex angle YOP or, $\frac{3\pi}{2}+\theta$. Since phase is not altered by a change of 2π, (Art. 43), this phase of N may be looked upon as $\frac{3\pi}{2}+\theta-2\pi$, or, $\theta-\frac{\pi}{2}$.

Thus, the phase of N is *behind* the phase of M by $\frac{\pi}{2}$.

The displacement equation of N is given by

$$y=a\cos\left(\theta-\frac{\pi}{2}\right) \quad \text{or,} \quad y=a\sin\theta.$$

Hence, a uniform circular motion is equivalent to two *S.H M.*'s executed in rectangular directions having the same amplitude and period but differing in phase by $\frac{\pi}{2}$.

60. Superposition of two equal and opposite circular motions—Let two points P_1 and P_2 move uniformly on the circle $XYX'Y'$ (fig. 20) starting simultaneously from the point X moving in the anti-clockwise and clockwise directions respectively with the same speed. Let at any time t, P_1 and P_2 be their respective positions on the circle. Since P_1 and P_2 move with the same speed starting simultaneously from X, the angles, P_1OX and P_2OX subtended by P_1 and P_2 at the centre O are equal. Hence, the straight line P_1MP_2 joining P_1 and P_2 is perpendicular to XOX' at M. Draw P_1N_1 and P_2N_2 perpendiculars to YOY'. If ω be the angular velocity of either of P_1 or P_2 on the circle, each of the angles P_1OX and P_2OX is ωt. By Art. 59 the circular motion of P_1 is equivalent to the *S. H. M.*'s of M and N_1 on XOX and YOY' respectively. The displacements of M and N_1 at the time t are

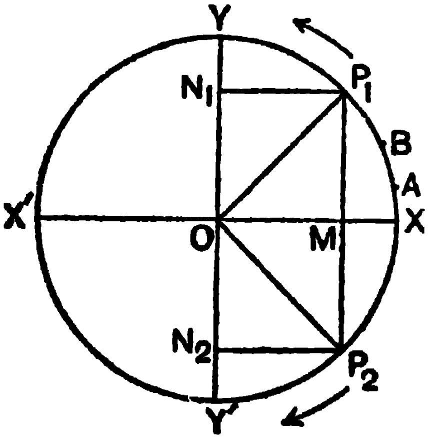


Fig. 20

$$x_1=a\cos\omega t$$
$$y_1=a\sin\omega t.$$

Similarly, the circular motion of P_2 is equivalent to the *S.H.M.*'s of M and N_2 on XOX' and YOY'. The displacements of M and N_2 at the time t are

$$x_2=a\cos\omega t$$
$$y_2=-a\sin\omega t.$$

If the two circular motions are communicated simultaneously to a point, its displacements x and y along XOX' and YOY' will be given by

$$x=x_1+x_2=2a\cos\omega t$$

and $$y=y_1+y_2=0.$$

The two circular motions are thus equivalent to an *S.H.M.* executed along the line *XOX'* with the same period as that of either circular motion.

If the two circular motions started from a point *A* (fig. 20) on the circle they would have been equivalent to an *S.H.M.* of the same period but executed along the diameter of the circle through *A*.

Hence, two equal and opposite circular motions of the same period are equivalent to an *S.H.M.* having its period equal to that of either circular motion executed along the diameter of the circle drawn through the point where the two circular motions meet.

Let us next suppose that the periods of the two circular motions are not exactly equal but the period of the anti-clockwise circular motion of P_1 is slightly less than that of the clockwise circular motion of P_2. Starting from the same point *X* on the circle and rotating round the circle, P_1 reaches the point *X* earlier than the point P_2. Hence, the two points P_1 and P_2 will meet at a point *A* slightly distant from *X* towards *Y*. This, following the deduction of this article, is equivalent to an *S.H.M.* along the diameter through *A* of the circle. After starting from *A* the two points P_1 and P_2 on completing their second rotation will meet at a point *B*, a little farther from *A* towards *Y*. Hence, they are equivalent to an *S.H.M.* executed along a diameter through *B*. Therefore, the two circular motions executed in opposite directions in periods which are not exactly equal are equivalent to an *S. H. M.* executed along a straight line which slowly rotates in the direction of the quicker circular motion.

61. Lissajous' Figures—The curves representing the resultant of two rectangular *S.H.M.*'s were for the first time produced by Lissajous by an optical method and hence these are known as Lissajous' figures. A few mechanical and optical methods of obtaining the curves are given below :

(*a*) **Blackburn's Pendulum**—It consists of a rectangular framework of wood fixed in a vertical plane (fig. 21) on a massive base. Two strings, wound round a screw peg *P* at their ends pass through the corners *A* and *B* of the framework. The strings are clamped jointly at a point *C*, by a clamp which can slide up and down. The other ends of the joined strings branch out in three portions which carry a heavy ring *M*. The arrangement combines in itself two pendulums.—if pulled in the plane of the framework and then let go, the ring executes an *S. H. M.* being the bob of a simple pendulum of length *CM*, where *M* is the middle point of the ring. If pulled perpendicular to the framework and released, the ring executes an *S H. M.* as it forms the bob of a simple pendulum of length *DM*, the point *D* being on the prolongation of *MC* where it meets *AB*. If it is required to combine two *S. H. M*'s of periods 2 : 3 the lengths *CM* and *DM* are to be in the ratio 4 : 9, as $T \propto \sqrt{l}$ in a simple pendulum, the laws of which are only approximately obeyed in the present case. For recording the motion, a glass funnel containing fine silver sand is placed within the hollow of the ring. Pulling the ring in an intermediate direction and then releasing it, the resultant curve is marked out by the fall of sand on a paper placed below the funnel By suitably adjusting the lengths *CM* and *DM* with the help of the clamp, the ratio of the periods can be altered to any desired value and the resultant curve can be drawn.

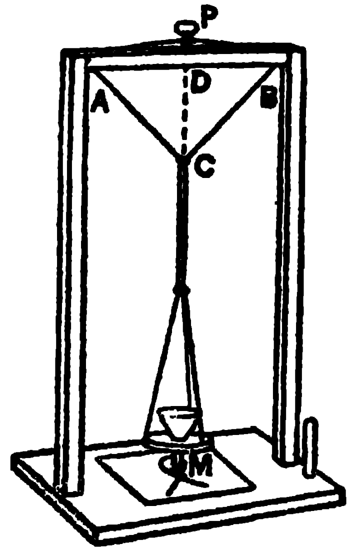


Fig. 21

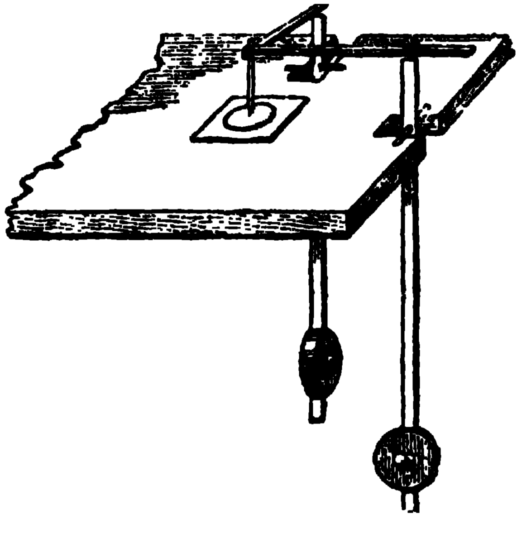


Fig. 22

(*b*) **Tisley's Harmonograph**—In one form of the apparatus, it consists of two heavy weights clamped near the ends of two iron rods (fig 22). The rods are provided with knife-edges about which they can oscillate. The pendulums thus formed are suspended by their knife-edges from the adjacent sides of a rectangular table. The rods project a little up the table and to the upper end of each a straight strip of iron is pivoted. The free ends of the two strips are hinged together, so as to include a right-angle and through their junction passes a fine pencil-point vertically. The weights execute *S. H. M.*'s in perpendicular directions, their resultant motion is drawn by the pencil point on a paper fixed on the table. The pendulums, however, do not obey the laws of simple pendulum, but by stop-watch the positions of the weights can be adjusted to give a particular ratio between their periods of vibration.

(*c*) **Kaleidophone**—The Kaleidophone, devised by Wheatstone (fig. 23) in 1827, consists of a straight rod of steel of rectangular cross-section. The rod is clamped on a massive base and a small polished bead is fixed on its free end. Because of the difference in widths of its cross-section in two rectangular directions, different elastic forces will be called into play, when the bead is displaced in two perpendicular directions. This will cause a difference in the periods of vibration in the two directions. There are a number of such rods with different ratios between the two periods of vibration of each. The motion of the bead shows the resultant curve.

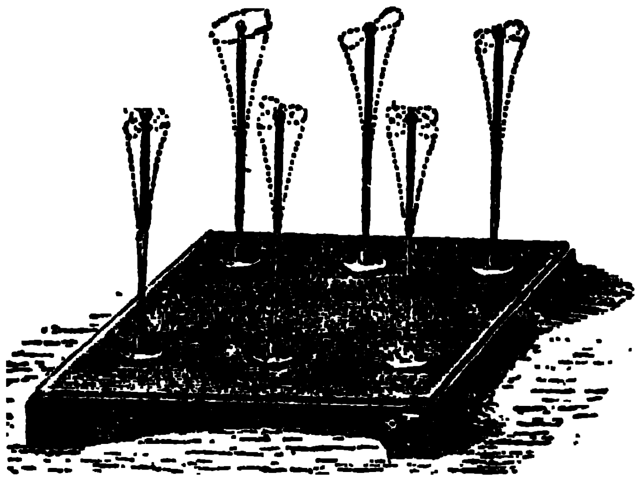


Fig. 23

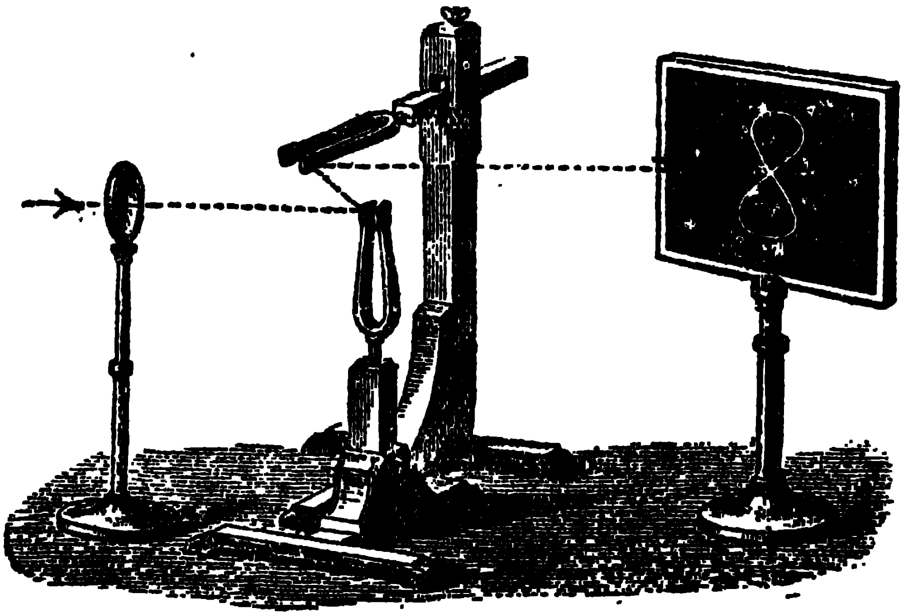


Fig. 24

(*d*) **Optical Method**--Lissajous obtained the curves by compounding the vibrations of two tuning-forks vibrating in two perpendicular directions. On one prong of each fork a light mirror was fixed (fig. 24). A narrow beam of light passed through a

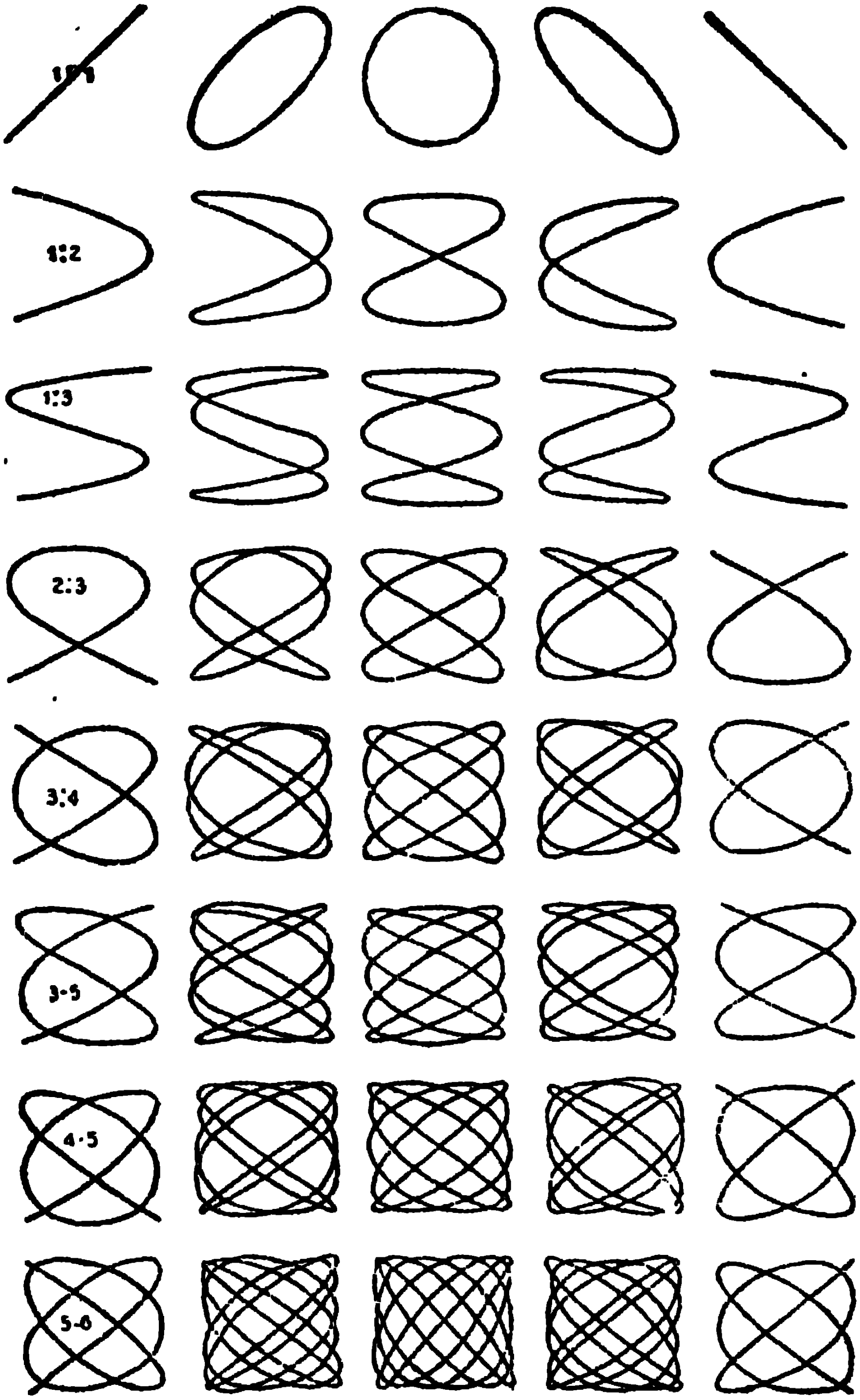


Fig. 25

focussing lens and was successively reflected at the two mirrors. The beam was finally focussed on a screen. The resultant of the two *S.H.M.*'s executed by the tuning-forks was drawn on the screen. Choosing two forks of suitable frequencies, a particular ratio between the periods of oscillation was introduced.

Recently Wood (1923)* has obtained Lissajous' figures at a frequency of the order of 10^6 vibrations per second by means of cathode-ray oscillograph.

Fig. 25 shows some of Lissajous' curves with different periods and initial phase-differences :—

Examples

1. What are Lissajous' figures ? Calculate the resultant of two rectangular vibrations whose amplitudes and periods are in the ratio of 1 : 2, the phase difference being 90°.

(*R. U. 1948, A. U. 1948*)

Let the amplitudes of the *S.H.M.* s along x and y directions be a and $2a$ respectively. Their periods also being in the ratio 1 : 2, when the point on the circle of reference of the 1st *S.H.M.* subtends θ, the other point will subtend $\theta/2$. With any initial phase difference of $\pi/2$, the motions can be represented by the equations

$$x=a\cos\theta \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

$$\text{and}\quad y=2a\cos\left(\frac{\theta}{2}+\frac{\pi}{2}\right)=-2a\sin\frac{\theta}{2} \qquad \ldots \qquad \ldots \qquad (2)$$

$$\text{From (2)},\ y^2=4a^2\sin^2\frac{\theta}{2}$$

$$\text{or, } y^2=2a^2(1-\cos\theta) \qquad \ldots \qquad \ldots \qquad (3)$$

If θ is eliminated between the two equations (1) & (3)

$$y^2+2ax=2a^2$$

$$\text{or, } y^2=2a(a-x)$$

The resultant motion is thus represented by a parabola symmetrical about x-axis with its vertex at $(a, 0)$.

2. A vertical *U*-tube of uniform cross-section contains water upto a height of 30 cms. Show that if the water on one side is depressed and released, its motion up and down the two sides of the tube is simple hamonic, and calculate its period. (*Delhi 1947*)

Let at any instant of time x cms. be the depression of the water level in one limb of the tube from the position of rest, the water level in the other limb will be elevated by the same height x cms. Hence the difference between the levels of water in the two limbs is $2x$ cms. The weight of water in this height $2x$ cms. provides the moving force on the water contained in the whole U-tube. If a sq.cm. be the cross-section of the U-tube, the moving force is thus $a\times 2x\times\rho.\,g$ dynes where ρ is the density of water in gms. per c.c. and g the acceleration of gravity. The mass of water in motion is $a\times 2\times 30\times\rho$ gms. By 2nd law, the

acceleration of the moving water $=\dfrac{a\times 2x\times\rho g}{a\times 2\times 30\times\rho}=\dfrac{gx}{30}$ cm. per sec. per sec.

The acceleration is thus proportional to the displacement x and is also directed oppositely to it. Hence, the motion of water is simple harmonic. The period T is given by

$$T=2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}=2\pi\sqrt{\frac{x}{gx/30}}$$

$$=2\pi\sqrt{\frac{30}{g}}=2\pi\sqrt{\frac{30}{981}}=1\cdot1 \text{ sec. nearly.}$$

3. **Vertical Oscillations of a Body Suspended by an Elastic String**—Let an elastic string of length l be suspended from a rigid support (fig. 26). On suspending a body of mass M, let its length be stretched by an amount a. The body is now under the action of two forces,—

(1) its weight Mg acting vertically downwards and

(2) the tension produced in the string acting vertically upwards.

If the extension of the string is within its elastic limit, following Hooke's law stress $\propto$ strain (Chap. IX), this latter force is proportional to the strain $\frac{a}{l}$ of the string. Representing the latter force by $\lambda\frac{a}{l}$ where λ is a constant, we have, since the two forces balance,

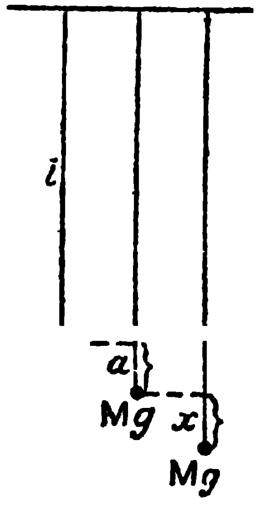


Fig. 26

$$\lambda\frac{a}{l}=Mg \qquad \dots \qquad \dots \qquad \dots \qquad (1)$$

Let the body be subsequently pulled down through a small extent and then released. The body will be executing oscillations in a vertical direction. At any time, let x be a further extension of the string. The downward force Mg on the body remains unaltered. The vertical upward force due to its tension, now becomes $\lambda\frac{a+x}{l}$, (supposing all these extensions are within the elastic limit).

Therefore, the resultant force on the body acting vertically upwards

$$=\lambda\frac{a+x}{l}-Mg$$

$$=\lambda\frac{a+x}{l}-\lambda\frac{a}{l} \qquad \ldots \qquad \ldots \qquad \text{[from (1)]}$$

$$=\lambda\frac{x}{l}$$

Hence the acceleration of the body $=\lambda\frac{x}{Ml}$

The acceleration is proportional to the displacement of the body and is directed oppositely to the displacement. The motion of the body is thus simple harmonic. Its period of oscillation is given by

$$T=2\pi\sqrt{\frac{Ml}{\lambda}} \qquad \ldots \qquad \ldots \qquad \text{(Art. 47)}$$

Substituting for λ from (1)

$$T=2\pi\sqrt{\frac{a}{g}}$$

The period of vertical oscillation of the body is thus equal to the period of oscillation of a simple pendulum of length a, the amount by which the body stretches the string.

Exercise

1. Explain the terms 'periodic motion' and 'simple harmonic motion'.

Find graphically the resultant of two simple harmonic motions at right-angles to each other, which have the same period and amplitude with a phase difference $\pi/4$. [*C. U. 1928, '36*]

2. Two simple harmonic oscillations of equal frequency and amplitude but in perpendicular directions are compounded. Find the resulting motion graphically. [*C. U. 1932*]

3. Deduce an equation for the S. H. motion of a particle, explaining the terms period, phase and epoch.

Two S. H. motions of the same period, but of different amplitudes and phases are acting in the same direction on a particle. Show that the resultant motion is simple harmonic, [*C. U. 1934*]

4. Explain clearly what you understand by simple harmonic motion and under what conditions a particle will execute such a motion.

Two simple harmonic motions of the same amplitude and period, but differing by $\pi/2$ in phase, are acting on a particle along two directions at right angles. Show graphically or mathematically that the resultant motion of the particle is circular. [*C. U. 1938*]

5. Two simple harmonic motions having the same period but differing in phase and amplitude are acting in the same direction on a particle. Show that the resultant motion is simple harmonic, and deduce the expression for the resultant amplitude and phase. [*C. U. 1940*]

6. Define the terms amplitude, period and phase, as applied to simple harmonic motion. Find graphically, or otherwise the resultant of two simple harmonic motions having the same period and amplitude but differing in phase by $\pi/4$ and acting at right angles to each other. Indicate a method of experimentally demonstrating the result obtained. [*C. U. 1946*]

7. Define an S. H. motion explaining the meanings of the terms *period amplitude* and *phase*.

A particle is subjected simultaneously to two S. H. vibrations of same period but of different amplitudes and phases in perpendicular directions. Find an expression for the resultant motion and show that the path traced by the particle is an ellipse. For what conditions, the path may be a circle and a straight line ? [*C. U. 1950*]

8. Define an S. H. motion and state under what conditions a particle will execute such a motion. Show that a uniform motion in a circle is equivalent to two S. H. motions at right-angles to each other [*C. U. 1953*]

9 What are Lissajous' figures ? Calculate the resultant of two rectangular vibrations whose amplitudes as well as periods are in the ratio of 1 : 2, the phase difference being 90°. [*R. U. 1948. A. U. 1948*]

10. What are Lissajous' figures ? Explain how with their help the frequencies of two tuning-forks can be compared. [*P. U. & E. P. U. 1951*]

11. Write notes on Lissajous' figures. [*N. U. 1950, L. U. 1951*]

12. Discuss the resultant motion when two simple harmonic vibrations of the same frequency but in directions at right angles to each other are compounded together. [*A. U. 1946*]

CHAPTER VI

MOMENT OF INERTIA

62. Rectilinear Motion—If a body moves *along* a straight line, the motion of the body is called *rectilinear* or *translatory*. The kinetic energy of the body, at any stage is obtained by the half of the product of its mass and square of its velocity.

63. Rotational Motion—When a body moves *about* a fixed straight line such that the different particles of the body are moving in circles round the fixed straight line, the motion of the body is called *rotational*. The fixed straight line is called the *axis of rotation*.

64. Energy of Rotation : Moment of Inertia—If a particle of mass m moves uniformly in a circle of radius r and if the angular velocity of the particle is ω, its linear velocity is of magnitude ωr and its kinetic energy of rotation is $\frac{1}{2}m\omega^2r^2$.

A rigid body is one whose parts always maintain an invariable position with respect to one another. It may be supposed to be built up of elements or particles of mass so small, that the distances between the different parts of a particle are negligibly small compared to the linear dimension of the rigid body. If a rigid body rotates uniformly round a fixed straight line, the different particles of it possess different kinetic energies of rotation; because, although the angular velocity of the particles is the same, still they are situated at different distances from the fixed straight line and hence their linear velocities are of different magnitude. To calculate the kinetic energy of rotation of the rigid body, we shall have to find the kinetic energy of each such particle and then sum up the kinetic energies of all the particles of which the rigid body is made. The result of this summation gives the kinetic energy of rotation of the rigid body.

To state mathematically, let a rigid body, rotating uniformly round a fixed straight line with angular velocity ω be made up of particles of masses m_1, m_2, m_3, etc., at distances r_1, r_2, r_3, etc., from the axis of rotation. The kinetic energy of rotation of the rigid body will be given by

$$E=\tfrac{1}{2}m_1\omega^2r_1^2+\tfrac{1}{2}m_2\omega^2r_2^2+\tfrac{1}{2}m_3\omega^2r_3^2+\text{etc.}$$

since ω is the same for all the particles,

$$\therefore\quad E=\tfrac{1}{2}\omega^2(m_1r_1^2+m_2r_2^2+m_3r_3^2+\text{etc.})$$

The quantity within brackets is sometimes represented by Σmr^2 and is called moment of inertia of the body about the fixed straight line. In words, if a body is supposed to be built up of particles, the sum of the products of the mass of each particle into the square of its distance from a fixed straight line is called the moment of inertia of the body *about* the fixed straight line as axis.

Thus, If I represents the moment of inertia of a body about an axis round which it rotates with an angular velocity ω, its kinetic energy of rotation

$$E=\tfrac{1}{2}I\omega^2.$$

In the simplest case, the moment of inertia of a particle of mass M about an axis at a distance r from the particle is Mr^2.

65. Physical Significance of Moment of Inertia—Newton's first law of motion tells us that a body at rest will not move or a body in uniform motion along a straight line will not stop or change its motion without the application of an external force causing it to do so. This property of the body is called *inertia* (of translatory motion). Again to produce a particular change in its state of motion the force necessary will be greater, greater the mass of the body. Thus, the mass of a body gives it the property of inertia or reluctance to translatory motion and is a measure for it.

In a similar manner, a body capable of rotation about an axis will not rotate or a body in uniform rotation about a straight line will not stop or change its rotation without the application of a couple causing it to do so. This property of the body is called its *inertia of rotation* or *rotational inertia.* Again to produce a particular change in its state of rotation the couple necessary will be greater, greater the mass of the body and also greater the distances of its particles from the axis of rotation. These distances are equivalent to a single 'effective distance' K, called the *radius of gyration of the body about the axis of rotation* (Art. 69). The product of mass M of the body and K^2 provides *rotational inertia* or reluctance to rotational motion and is called the *moment of inertia of the body about the axis of rotation.* It plays an analogous role in rotational motion as mass does in rectilinear motion.

66. Moment of Momentum—Let a particle of mass m move about a straight line in a circle of radius r. If ω be the angular velocity of the particle, its linear speed$=\omega r$

Momentum of the particle$=m\omega r$

The moment of this momentum about the straight line$=m\omega r^2$

If instead of particle, a body of definite shape and size move in a circle about a straight line, its moment of momentum about the straight line will be calculated by summing up the moments of momentum of the particles into which the body may be supposed to be divided. Thus, if m_1, m_2, m_3, etc., be the masses of the particles of the body situated at distances r_1, r_2, r_3, etc., from the straight line about which the body is rotating, the moment of momentum of the body about the straight line

$$=m_1\omega r_1^2+m_2\omega r_2^2+m_3\omega r_3^2+\ldots$$

$$=\omega(m_1r_1^2+m_2r_2^2+m_3r_3^2+\ldots)$$

$=I\omega$ where I is the moment of inertia of the body about the straight line as axis.

The moment of momentum of a body is also known as *angular momentum*.

67. **Conservation of Angular Momentum**—Corresponding to the law of conservation of momentum for rectilinear motion (Art. 12) there is a law of conservation of angular momentum for bodies in rotational motion. It states that the angular momentum of a body rotating about an axis remains constant, if no *external* force is applied to it.

This is readily observed when a stone tied to one end of a string is whirled round by holding the other end of the string. If the application of force by the hand is stopped, the string winds itself round the hand with continuously increasing velocity. As the distance of the stone from the hand decreases, its moment of inertia about the axis of rotation gradually decreases. The angular velocity of the stone gradually increases so that its angular momentum $I\omega$ is conserved.

Let a man hold two weights in his two hands and stand on a smooth turn-table with hands fully stretched horizontally and the turn-table set to uniform rotation. On drawing the hands towards his shoulders, he will feel that the angular velocity of rotation has increased. Drawing the hands towards the shoulders, the moment of inertia of the system about the axis of rotation is decreased with a consequent increase in the angular velocity to an amount consistent with the conservation of angular momentum.

68. **Analogy between Rectilinear and Rotational Motions**—An analogy can be deduced between rectilinear and rotational motions. The kinetic energy for rectilinear motion is given by $\frac{1}{2}Mv^2$, whereas that for rotational motion is $\frac{1}{2}I\omega^2$. Thus, the mass and linear velocity in rectilinear motion are analogous to the moment of inertia and angular velocity respectively in rotational motion.

A uniformly accelerated rectilinear motion is produced by a force, the acceleration of the motion is obtained from the second law of motion, *viz.*,

$$\text{Linear acceleration} = \frac{\text{Moving force}}{\text{Mass moved}}$$

A uniformly accelerated angular motion is produced by a couple, *i.e.*, two equal, parallel and opposite forces whose lines of action are not the same. The turning effect of a couple is measured by its moment which is known as the *torque*. Thus, the law analogous to the second law of motion in rotational dynamics is,

$$\text{Angular acceleration} = \frac{\text{Applied torque}}{\text{Moment of inertia about the axis of rotation}}$$

The following table shows the relation between rectilinear and rotational dynamics.

Rectilinear motion	Rotational motion
Displacement (linear), x	Displacement (angular), θ
Velocity ,, $\frac{dx}{dt}, v$	Velocity ,, $\frac{d\theta}{dt}, \omega$
Acceleration ,, $\frac{d^2x}{dt^2}, \frac{dv}{dt}$	Acceleration ,, $\frac{d^2\theta}{dt^2}, \frac{d\omega}{dt}$
Mass, M	Moment of inertia, I
Force, $M\frac{d^2x}{dt^2}$	Torque, $I\frac{d^2\theta}{dt^2}$
Kinetic energy (linear), $\frac{1}{2}Mv^2$	Kinetic energy (angular), $\frac{1}{2}I\omega^2$
Momentum (linear), Mv	Momentum (angular), $I\omega$

69. Radius of Gyration—It has been observed in Art. 64, that if a body of mass M is supposed to be built up of particles of masses m_1, m_2, m_3, etc., situated at distances r_1, r_2, r_3, etc., from a particular axis, the quantity

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\ldots = \Sigma mr^2$$

is the moment of inertia of the body about the axis.

Let $\Sigma mr^2 = MK^2$

where $M = \Sigma m$ is the mass of the body and K is of the dimension of length. Again, MK^2 represents the moment of inertia of a particle of mass M about the same axis at a distance K from it. (Art. 64). The quantity K is called the *radius of gyration* of the body about the axis. In words, 'if the mass of a body were concentrated at a point so as to possess the same moment of inertia as the body about an axis, the distance of the point from the axis is called the '*radius of gyration* of the body about the axis.' The radius of gyration is sometimes called the *swing radius*.

In the simplest case, the radius of gyration of a particle of mass about an axis at a distance r from it is equal to r.

70. Energy of a Rolling Wheel—The kinetic energy of a wheel which moves along a straight line while rolling, arises due to (1) motion of rotation and (2) motion of translation.

If I be the moment of inertia of the wheel about its axle and ω its angular velocity, its kinetic energy of rotation

$$=\tfrac{1}{2}I\omega^2 \text{ (Art. 64)}$$

Again, if k be the radius of gyration of the wheel about the axle, $I=Mk^2$ where M=mass of the wheel.

$\therefore$ kinetic energy of rotation$=\frac{1}{2}Mk^2\omega^2$.

Also if R be the radius of the wheel and v, the linear velocity of the centre of the wheel (which is the velocity of the carriage to which the wheel is attached).

$$\omega=\frac{v}{R}$$

$$\therefore \text{ kinetic energy of rotation}=\frac{1}{2}\frac{Mk^2v^2}{R^2}. \qquad \ldots (1)$$

$$\text{The kinetic energy of translatory motion}=\tfrac{1}{2}Mv^2 \qquad \ldots (2)$$

The total kinetic energy of the wheel is the sum of (1) and (2)

$\therefore$ kinetic energy of the wheel

$$\frac{1}{2}\frac{Mk^2v^2}{R^2}+\frac{1}{2}Mv^2$$

$$=\frac{1}{2}Mv^2\left(\frac{k^2}{R^2}+1\right)$$

71. Theorem of Perpendicular Axes. *Laminar body*—Consider a lamina (thin plate) and two axes OX and OY (*fig.* 27) at right-angles to each other in the plane of lamina meeting at O. The theorem states that the moment of inertia of the lamina about an axis OZ, perpendicular to the plane of the lamina passing through O, will be the sum of the moments of inertia of the lamina about the axes OX and OY. Representing the moments of inertia about OX, OY and OZ, by I_x, I_y, and I_z respectively, the theorem states that $I_z=I_x+I_y$.

Proof—Consider a particle of mass m at P having co-ordinates x, y and z. Then the distances of P from the axes OX, OY and OZ are y, x and z respectively. Using symbols of Art. 69.

$$I_x = \Sigma my^2, \quad I_y = \Sigma mx^2$$

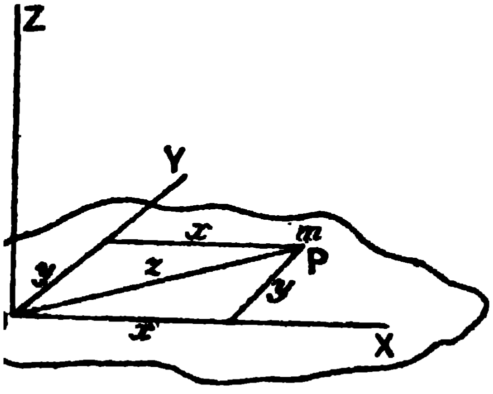


Fig. 27

$$\therefore \quad I_x + I_y = \Sigma my^2 + \Sigma mx^2 = \Sigma m(y^2 + x^2)$$
$$= \Sigma mz^2 = I_z \text{ (as } y^2 + x^2 = z^2)$$

This proves the theorem.

72. Theorem of Parallel Axes. *Laminar body*—Consider a lamina and two axes AB and CD (fig. 28) in the plane of the lamina parallel to each other, of which CD passes through the centre of mass G of the lamina. The theorem states that the moment of inertia of the lamina about the axis AB will be the sum of the moment of inertia of the lamina about the axis CD and the product of the mass of the lamina into the square of the distance between the two axes. Representing the moments of inertia about AB and CD by I_{AB} and I_{CD} respectively, the mass of the lamina by M and the distance between AB and CD by h, the theorem states $I_{AB} = I_{CD} + Mh^2$.

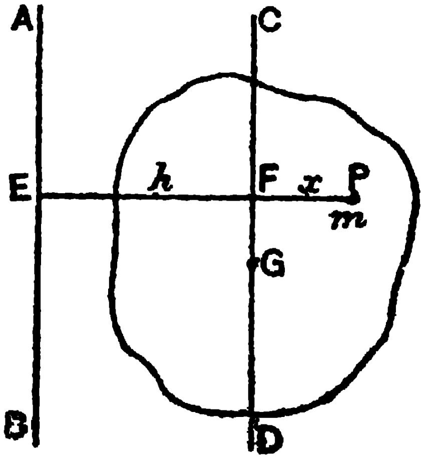


Fig. 28

Proof—Consider a particle of mass m at P. Draw a perpendicular PFE to CD and AB meeting them in F and E respectively. Let $FP = x$.

The moment of inertia of the lamina about the axis AB is given by

$$I_{AB} = \Sigma m(x+h)^2$$
$$= \Sigma m(x^2 + h^2 + 2xh)$$
$$= \Sigma mx^2 + \Sigma mh^2 + \Sigma 2mxh$$
$$= \Sigma mx^2 + h^2 \Sigma m + 2h \Sigma mx \quad \text{(since } h \text{ is a constant).}$$

The first term Σmu^2 of the expression represents the moment of inertia of the lamina about the axis CD. The second term $h^2\Sigma m = Mh^2$, where M is the mass of the lamina. The third term represents $2h\times$the moment of mass of the lamina about CD. Since CD passes through the centre of mass of the lamina, the lamina will balance about a knife-edge below CD. Therefore, the moment of mass of the lamina with reference to CD is zero. Hence, we have

$$I_{AB} = I_{CD} + Mh^2.$$

This proves the theorem.*

73. Calculation of the Moment of Inertia for Particular Cases—

(1) Thin uniform rod about one end.

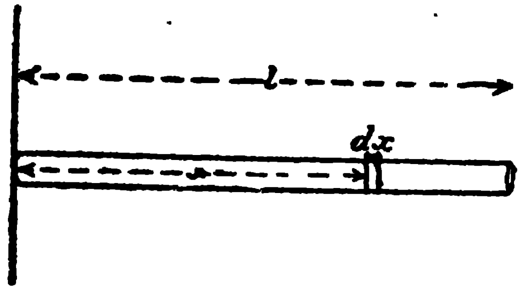


Fig. 29

Consider a thin uniform rod of length l (fig. 29) and an axis perpendicular to its length passing through an end, and consider a short element of length dx at a distance x from the axis. The rod being thin, the parts of the element may be supposed to be at the same distance x from where the rod meets the axis.

If the mass of the rod be M, the mass of the element $=\frac{M}{l}dx$ and the moment of inertia of the element about the axis $=\frac{M}{l}dx.\,x^2$ (Art. 64).

Integrating this expression between the limits $x=0$. to $x=l$, the moment of inertia of the rod

$$= \int_0^l \frac{M}{l}x^2\,dx = \frac{M}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{1}{3}Ml^2$$

The radius of gyration about the axis$=l/\sqrt{3}$

---

* The theorems of the perpendicular and parallel axes also hold good for three dimensional bodies.

(2) Thin uniform rod about an axis through its centre and perpendicular to its length.

Consider a thin rod of a mass M, length l (fig.30) and an axis through its centre perpendicular to its length, and consider an element of length dx at a distance x from the axis.

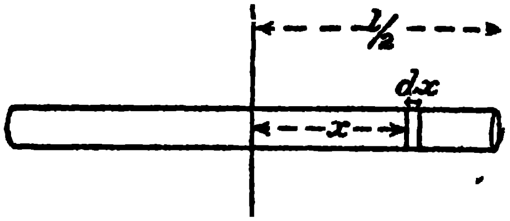


Fig. 30

From (1) above, the moment of inertia of the element about the axis

$$=\frac{M}{l}dx.x^2 \qquad \text{(Art. 64)}$$

Integrating the expression between $x=0$ to $x=\frac{l}{2}$, the moment of inertia of one-half of the rod

$$=\int^{l/2}\frac{M}{l}x^2dx=\frac{M}{l}\left[\frac{x^3}{3}\right]^{l/2}=\frac{1}{24}Ml^2$$

Hence, the moment of inertia of the rod about the axis $=\frac{1}{12}Ml^2$*

The radius of gyration about the axis$=l/\sqrt{12}$

(3) Moment of inertia of a rectangular lamina about an axis through its centre of mass parallel to one of its sides.

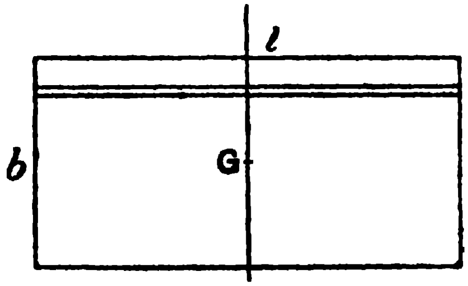


Fig. 31

Consider a rectangular lamina, the length and breadth of which are respectively l and b. The axis is parallel to the side b and passes through the centre of mass G of the rectangle (fig. 31). Imagine the rectangle to be divided into equal narrow strips parallel to its length. Let m be the mass of one of these strips.

The moment of inertia of the strip about the axis by (2) above $=\frac{1}{12}ml^2$

---

*The theorem can also be proved from (1) above by applying the theorem of parallel axes.

The moment of inertia of the rectangular lamina is the sum of the moments of inertia of all the strips into which the rectangle is divided and is equal to

$$\Sigma\frac{1}{12}ml^2=\frac{1}{12}l^2\Sigma m=\frac{1}{12}Ml^2$$

where $\Sigma m=M$, the mass of the rectangular lamina.

Similarly, the moment of inertia of the lamina about an axis through its centre of mass parallel to the side l

$$=\frac{1}{12}Mb^2.$$

By the theorem of perpendicular axes the moment of inertia of the lamina about an axis passing through its centre of mass and perpendicular to the plane of the lamina

$$=\frac{M}{12}(l^2+b^2)$$

The radius of gyration about the axis through G perpendicular to its plane $=\sqrt{(l^2+b^2)/12}$.

Cor. Moment of inertia of a rectangular bar having a, b and c for its length, breadth and thickness, about an axis passing through its centre of mass and parallel to the side of length c

$$=M\frac{a^2+b^2}{12}.$$ where M=mass of the bar.

Because, the bar may be supposed to be divided into rectangular laminæ of equal thickness having a and b for its sides. The moment of inertia of each lamina about an axis through its centre of mass and perpendicular to its plane (*i.e.*, parallel to the side c of the bar)

$$=m\frac{a^2+b^2}{12}$$ where m=mass of the lamina

Hence, the moment of inertia of the bar about an axis through its centre of mass and parallel to side c

$$=\Sigma m\frac{a^2+b^2}{12}=M\frac{a^2+b^2}{12}$$

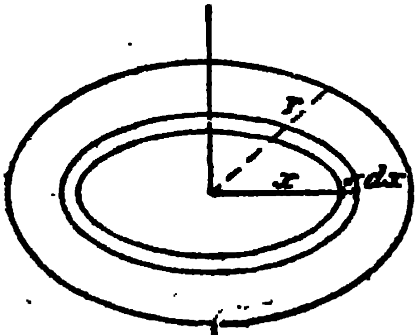


Fig. 32

(4) Moment of inertia of a circular disc about an axis passing through its centre and perpendicular to its plane.

Consider a disc of radius r and mass M, and consider a narrow concentric circular ring of radius x and width dx (fig. 32). The area of the ring is evidently the product of its circumference and width, *i.e.*, $2\pi x\,dx$, The area of the disc $=\pi r^2$

$\therefore$ the mass of the ring $\frac{M}{\pi r^2}2\pi x dx=\frac{2M}{r^2}x dx$

The moment of inertia of the ring about the axis

$$=\frac{2M}{r^2}x dx . x^2=\frac{2M}{r^2}x^3 dx$$

The moment of inertia of the disc about the axis

$$=\int_0^r \frac{2M}{r^2}x^3 dx=\frac{2M}{r^2}\left[\frac{x^4}{4}\right]_0^r=\frac{1}{2} Mr^2$$

The radius of gyration about the axis $= r/\sqrt{2}$.

Cor. (*i*) *Moment of inertia of a circular disc about a diameter*—Let I be its moment of inertia about a diameter, then I is also its moment of inertia about a perpendicular diameter. Therefore, by the theorem of perpendicular axes the moment of inertia of the disc about an axis through its centre and perpendicular to its plane

$$=I+I=\frac{1}{2}Mr^2 \text{ [by (4) above]}$$

$$\therefore \quad I=\frac{1}{4}Mr^2$$

(*ii*) *Moment of inertia of a circular disc about a tangent.*

If I be the required moment of inertia, by the theorem of parallel axes and from above

$$I=\frac{1}{4}Mr^2+Mr^2=\frac{5}{4}Mr^2$$

(5) Moment of inertia of a flat ring about an axis through centre and perpendicular to its plane.

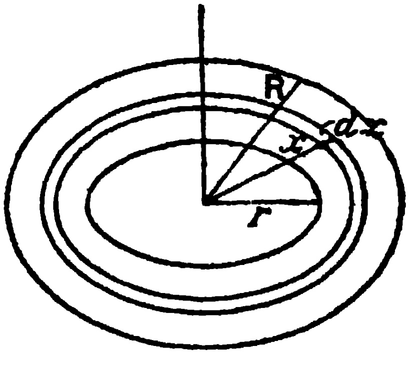


Fig. 33

A flat ring consists of a circular disc from which the central portion has been removed by a concentric circular cut. Let the internal and external radii of the ring be r and R respectively (fig. 33) and let m be the mass per unit area of the ring. As in (4), the area of a concentric circular strip having radius x and a small width dx is $2\pi x dx$. The mass of this strip is $2\pi m x dx$ and its moment of inertia about the axis is $2\pi m x dx . x^2$.

Hence, the moment of inertia of the given flat ring about the axis

$$=\int_r^R 2\pi m x dx . x^2 = 2\pi m \left[\frac{x^4}{4}\right]_r^R = \frac{1}{2}\pi m(R^4 - r^4)$$

$$=\frac{1}{2}\pi m(R^2 - r^2)(R^2 + r^2).$$

But $\pi(R^2 - r^2)$ is the surface area of the flat ring and $\pi m(R^2 - r^2)$ is its mass.

Therefore, the required moment of inertia $=\frac{1}{2}M(R^2+r^2)$ where M = the mass of the flat ring.

The radius of gyration about the axis $= \sqrt{(R^2+r^2)/2}$.

Cor. (*i*) *Moment of inertia of a flat ring about a diameter by corollary of* (4) *above*

$$=\frac{1}{4}M(R^2+r^2).$$

(*ii*) *Moment of inertia of a thin* (*of negligible width*) *circular ring about an axis through its centre and perpendicular to its plane.*

This is directly obtained from (5) above, putting $R=r$, the radius of the ring.

Hence, the required moment of inertia $=Mr^2$ where M = mass of the ring.

It can also be proved independently as follows :—

The moment of inertia of an element of mass m of the thin ring is mr^2. Summing for all such elements, the moment of inertia of the thin ring about the axis $=\Sigma mr^2 = Mr^2$.

(*iii*) *Momemt of inertia of a thin circular ring about a diameter by the corollary* (*i*) *of* (5) $=\frac{1}{2}Mr^2$.

(*iv*) By Cor. (*i*) and (*ii*) above and by the theorem of parallel axes.

Moment of inertia of a flat ring about a tangent

$$=\frac{1}{4}M(R^2+r^2)+MR^2=\frac{1}{4}M(5R^2+r^2).$$

Also, *Moment of inertia of a thin ring about a tangent*

$$=\frac{1}{2}Mr^2+Mr^2=\frac{3}{2}Mr^2.$$

(6) Moment of inertia of a circular cylinder about an axis though centre of mass and perpendicular to its length.

Let l, r and M represent the length, radius of the cross-section and the mass of the cylinder respectively. AB (fig. 34) is the axis about which its moment of inertia is to be found.

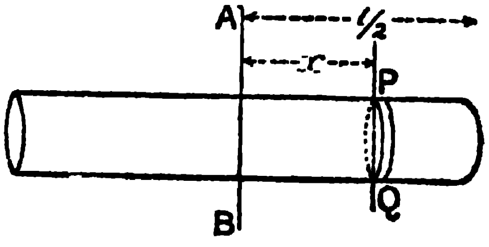


Fig. 34

Consider a *thin* disc PQ of thickness dx of the cylinder at a distance x from AB. The mass of the disc$=\frac{M}{l}dx$. The moment of inertia of the disc about a diameter PQ parallel to $AB=\frac{1}{4}\frac{M}{l}dxr^2$ (corollary of 4). Since the disc is thin PQ may be regarded as an axis passing through the centre of mass of the disc. By the principle of parallel axes the moment of inertia of the disc about AB

$$=\frac{1}{4}\frac{M}{l}dxr^2+\frac{M}{l}dx\,x^2$$

The moment of inertia of the cylinder about AB will be obtained by integrating the above expression between the limits o and $l/_2$ and then doubling it.

Hence, moment of inertia of the cylinder about AB

$$=2\int_o^{l/2}\left(\frac{1}{4}\frac{M}{l}r^2dx+\frac{M}{l}x^2dx\right)$$

$$=\frac{1}{2}\frac{M}{l}r^2\int_o^{l/2}dx+\frac{2M}{l}\int_o^{l/2}x^2dx$$

$$=\frac{1}{2}\frac{M}{l}r^2\cdot\frac{l}{2}+\frac{2M}{l}\frac{l^3}{24}$$

$$=M\frac{r^2}{4}+M\frac{l^2}{12}=M\left(\frac{l^2}{12}+\frac{r^2}{4}\right)$$

The radius of gyration about the axis$=\sqrt{\frac{l^2}{12}+\frac{r^2}{4}}$

Cor. If the cylinder is hollow having r and R for its internal and external radii, PQ (fig. 34) represents a flat ring of mass $\frac{M}{l}dx$ and its moment of inertia about a diameter perallel to AB $=\frac{1}{4}\,\frac{M}{l}(R^2+r^2)dx$ (from 5, cor. *i*) and its moment of inertia about AB by theorem of parallel axes

$$=\frac{1}{4}\,\frac{M}{l}(R^2+r^2)\,dx+\frac{M}{l}dx.x^2$$

∴ moment of inertia of the hollow cylinder about AB

$$=2\int_0^{l/2}\frac{1}{4}\,\frac{M}{l}(R^2+r^2)dx+2\int_0^{l/2}\frac{M}{l}x^2dx$$

$$=\frac{M}{4}(R^2+r^2)+\frac{M}{12}l^2=M\left(\frac{l^2}{12}+\frac{R^2+r^2}{4}\right)$$

7. Moment of inertia of a Circular Cylinder about its own axis.

Consider a circular cylinder having r for the radius of cross-section (fig. 35). Imagine the cylinder to be divided into thin discs perpendicular to its axis. Let m be the mass of one of these discs. The moment of inertia of a disc about the axis by (4) above $=\frac{1}{2}mr^2$

The moment of inertia of the cylinder is the sum of the moments of inertia of all the discs into which the cylinder is divided and is equal to

$$\Sigma\tfrac{1}{2}mr^2=\tfrac{1}{2}r^2\Sigma m=\tfrac{1}{2}Mr^2$$

as Σm represents the mass of all the discs and is equal to M, the mass of the cylinder.

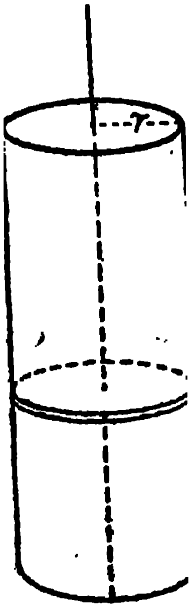


Fig. 35

The radius of gyration about the axis

$$=r/\sqrt{2}.$$

Cor. Moment of inertia of a cylindrical shell having internal and external radii r and R respectively about its axis is obtained by dividing it into flat rings. The moment of inertia of a ring of mass m by (5) above is

$$\tfrac{1}{2}m(R^2+r^2).$$

Hence, the moment of inertia of the shell of mass M about its axis

$$=\Sigma\tfrac{1}{2}m(R^2+r^2)=\tfrac{1}{2}M(R^2+r^2).$$

The radius of gyration about the axis $=\sqrt{\frac{R^2}{}+r^2}$

(5) Moment of inertia of solid sphere about a diameter.

Consider a sphere of mass M and radius r (fig. 36). Let AB be a diameter about which the moment of inertia of the sphere is to be found out.

Consider a thin circular slice CD of thickness dx of the sphere at a distance x from the centre O and having its plane perpendicular to the diameter AB. The radius of the slice

$$=\sqrt{r^2-x^2}$$

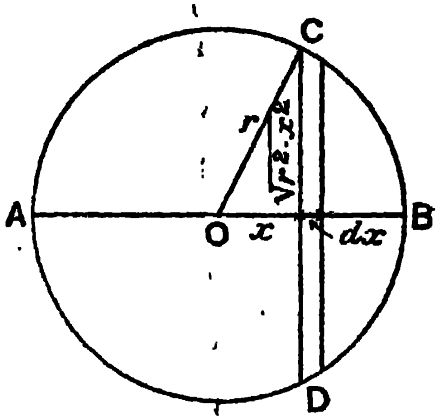


Fig. 36

The volume of the slice is evidently the product of its surface area and thickness, *i.e.*, $\pi(r^2-x^2)dx$

The volume of the sphere $=\frac{4}{3}\pi r^3$

Therefore the mass of the slice

$$=\frac{M}{\frac{4}{3}\pi r^3}\pi(r^2-x^2)dx=\frac{3M}{4r^3}(r^2-x^2)dx$$

The moment of inertia of the slice about the diameter AB, by (4) above

$$=\frac{1}{2}\cdot\frac{3M}{4r^3}(r^2-x^2)dx.\ (r^2-x^2)=\frac{3M}{8r^3}(r^2-x^2)^2dx$$

The moment of inertia of the sphere about the diameter

$$=2\int_{x=0}^{x=r}\frac{3M}{8r^3}(r^2-x^2)^2dx$$

$$=\frac{3M}{4r^3}\int_{x=0}^{x=r}(r^4-2r^2x^2+x^4)dx$$

$$=\frac{3M}{4r^3}\left[r^4x-\frac{2}{3}r^2x^3+\frac{x^5}{5}\right]_0^r$$

$$=\frac{3M}{4r^3}\left[r^5-\frac{2}{3}r^5+\frac{1}{5}r^5\right]$$

$$=\frac{3M}{4r^3}\cdot\frac{8r^5}{15}=\frac{2}{5}Mr^2$$

The radius of gyration about the axis$=\sqrt{\frac{2}{5}}r$

Cor. (*i*) *The moment of inertia of a solid sphere about a tangent* by the theorem of parallel axes.

$$=\frac{2}{5}Mr^2+Mr^2=\frac{7}{5}Mr^2$$

Cor, (ii) *Moment of inertia of a thick spherical shell having internal and external radii r and R about a diameter.*

Let I be the moment of inertia of the shell of mass M.

Moment of inertia of a solid sphere of radius R is the sum of the moment of inertia of a solid sphere of radius r and the moment of inertia of the thick spherical shell having internal and external radii r and R. Hence, if ρ is the density of the shell, from (8).

$$\frac{2}{5}\cdot\frac{4}{3}\pi R^3\rho.\,R^2=\frac{2}{5}\cdot\frac{4}{3}\pi r^3\rho.\,r^2+I$$

or, $I=\frac{8}{15}\pi\rho(R^5-r^5)$

Again, $\rho=\dfrac{M}{\frac{4}{3}\pi(R^3-r^3)}$ $\therefore\ I=\frac{2}{5}M\dfrac{R^5-r^5}{R^3-r^3}$

(9) Moment of Inertia of a Hollow Sphere about a diameter.

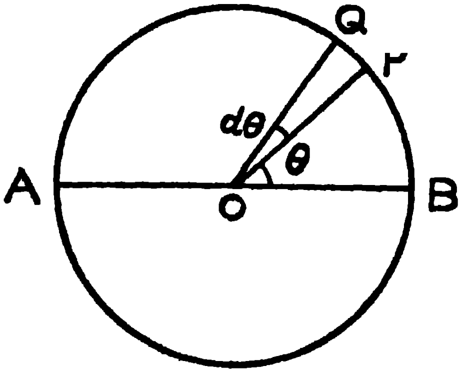


Fig. 37

Let AB (fig. 37) be a diameter and O the centre of the hollow sphere. Consider two neighbouring points P and Q on the sphere such that $\angle POB=\theta$, $\angle POQ=d\theta$.

If r be the radius of the sphere, the radius of the ring formed by the revolution of PQ about $AB=r\sin\theta$ and its width $PQ=rd\theta$.

Hence, the area of the ring$=2\pi r\sin\theta.rd\theta=2\pi r^2\sin\theta d\theta$. Mass of the ring $=\frac{M}{4\pi r^2}\cdot 2\pi r^2\sin\theta d\theta=\frac{1}{2}M\sin\theta d\theta$ where M=mass of the sphere.

Moment of inertia of the ring about the diameter AB (by cor. *ii* of 5) $=\frac{1}{2}M\sin\theta d\theta.\ r^2\sin^2\theta=\frac{1}{2}Mr^2\sin^3\theta d\theta$.

Moment of inertia of the sphere about the diameter AB

$$=\int_0^{\pi}\frac{1}{2}Mr^2\sin^3\theta d\theta=Mr^2\int_0^{\pi/2}\sin^3\theta d\theta=\frac{2}{3}Mr^2$$

This result can also be obtained from cor (ii) of (8)

The moment of inertia of a thick spherical shell of internal and external radii r and R.

$$=\frac{2}{5}M\frac{R^5-r^5}{R^3-r^3}=\frac{2}{5}M\frac{R^4+R^3r+R^2r^2+Rr^3+r^4}{R^2+Rr+r^2}$$

For a thin hollow sphere $R=r$

∴ Moment of inertia of a hollow sphere about a diameter

$$=\frac{2}{5}M\frac{5r^4}{3r^2}=\frac{2}{3}Mr^2$$

(10) Moment of Inertia of a Solid Cone about its own axis.

Consider a solid cone of mass M and height h, having R for the radius of its base (fig. 38). Imagine a thin slice of the cone perpendicular to its axis having thickness dx at a distance x from the apex of the cone; let r be its radius. Volume of the slice $=\pi r^2dx$. Vol. of the cone $=\frac{1}{3}\pi R^2h$.

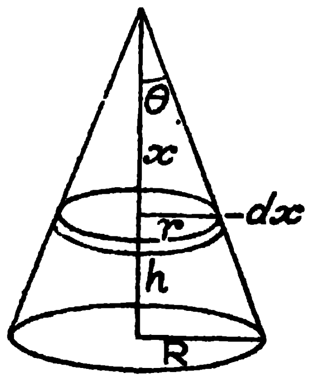


Fig. 38

Therefore, mass of the slice or disc

$$=\frac{M}{\frac{1}{3}\pi R^2h}\pi r^2dx=\frac{3M}{R^2h}r^2dx$$

Moment of inertia of the disc about the axis of the cone from (5)

$$=\frac{1}{2}\frac{3M}{R^2h}r^2dx.\ r^2=\frac{1}{2}\frac{3M}{R^2h}r^4dx$$

Again from similar triangles. $\frac{x}{r}=\frac{h}{R}$

$\therefore\ r=\frac{R}{h}x$ and $r^4=\frac{R^4}{h^4}x^4$

Substituting for r^4, the moment of inertia of the disc about the axis of the cone

$$=\frac{3MR^2}{2h^5}x^4dx$$

Moment of inertia of the cone about its own axis

$$=\frac{3MR^2}{2h^5}\int^{h} x^4dx=\frac{3}{10}MR^2$$

(11) Moment of inertia of a triangular lamina about an axis through its vertex perpendicular to its plane.

ABC represents a triangular lamina having sides of length a, b, c, (fig. 39). Let σ be the mass per unit area of the lamina. Consider a strip bc parallel to the side BC. Let p be the distance of BC, and x that of the strip bc from A, and dx, the width of the strip. Let also m be the length of the median from A on BC.

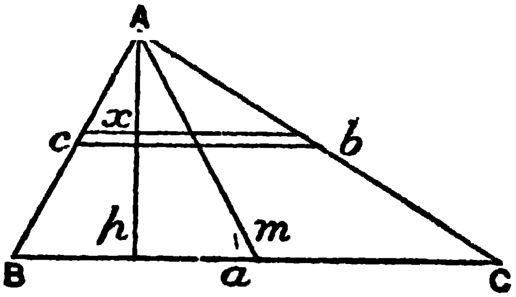


Fig. 39

The length of the strip $=\frac{ax}{p}$, its area $=\frac{axdx}{p}$ and its mass $=\frac{a}{p}\sigma xdx$.

Moment of inertia of the strip about an axis through its middle point perpendicular to its plane

$$=\frac{1}{12}\frac{a}{p}\sigma xdx\left(\frac{ax}{p}\right)^2 \quad \text{(from (2) above)}$$

Again, the distance between A and the middle point of the strip $=\frac{m}{p}x$

$\therefore$ moment of inertia of the strip about the given axis through A (by the theorem of parallel axes)

$$=\frac{1}{12}\frac{a}{p}\sigma x\,dx\left(\frac{ax}{p}\right)^2+\frac{a}{p}\sigma xdx\left(\frac{mx}{p}\right)^2$$

$$=\frac{1}{12}\frac{a\sigma}{p^3}(a^2+12m^2)x^3dx$$

Hence, moment of inertia of the triangular lamina about the axis

$$=\frac{1}{12}\frac{a\sigma}{p^3}(a^2+12m^2)\int_0^p x^2dx$$

$$=\frac{1}{24}ap\sigma\left(\frac{a^2}{2}+6m^2\right)$$

$$=\frac{M}{12}(8b^2+3c^2-a^2)$$

as M, the mass of the lamina $=\frac{ap\sigma}{2}$; and it can easily be shown that $m^2=\frac{b^2}{2}+\frac{c^2}{2}-\frac{a^2}{4}$.

(12) Moment of inertia of a triangular lamina about an axis through the vertex parallel to the base.

Let the axis be parallel to the side BC through A (fig. 89).

From (11) above, moment of inertia of the strip about the axis $=\frac{ax\sigma dx}{p}x^2$.

Hence, moment of inertia of the triangular lamina about the axis

$$=\int_p^p\frac{ax\sigma}{p}x^2dx=\tfrac{1}{4}a\sigma p^3=\tfrac{1}{2}Mp^2$$ as M, the mass of the lamina $=\frac{1}{2}ap\sigma$.

74. Routh's Rule—Routh's rule for finding the moment of inertia of symmetrical bodies is a very helpful one in many simple cases. The rule states, "When the axis about which the moment of inertia is required is an axis of symmetry passing through the centre of mass of the body, the moment of inertia in a large number of simple cases is given by $\text{Mass}\times\frac{\text{sum of squares of perpendicular semi-axes}}{3,\ 4\ \text{or}\ 5}$ where the denominator is to be taken 3, 4 or 5 according as the body is a rectangle, ellipse (including circle), or ellipsoid (including sphere)".

Thus, the moment of inertia of a solid sphere of mass M and radius r about a diameter

$$=M\frac{r^2+r^2}{5}=\frac{2}{5}Mr^2$$

75. Experimental Determination of Moment of Inertia—

(1) **Torsion Pendulum**—The moment of inertia of a body can easily be found by constructing a *torsion pendulum* with it and finding the period of oscillation of the torsion pendulum. If a wire is fixed at one end and twisted at the other end which is loaded, by the application of a couple, experiment shows that within certain limits, the torque required to produce a certain twist is proportional to the angle of twist. The torque required to produce a certain twist, however, depends upon the length, diameter and material of the wire*.

Let a body be suspended from one end of a wire, the other end of which is fixed. If the body is turned through a small angle and subsequently released, it will be set to torsional oscillations twisting the wire forwards and backwards. If at any instant of time θ be the angular displacement of the body from its position of rest, the moment of the couple or torque which tends to turn the body to its undisturbed position is proportional to θ.

Let the torque$=\tau\theta$ where τ is a constant depending upon the length, radius and material of the wire. It is called the *torque per unit twist* or the *torsional rigidity* of the wire. The angular acceleration of the body is given by (Art. 68.)

$$\frac{d^2\theta}{dt^2}=\frac{\text{Applied torque}}{\text{Moment of inertia of the system about the axis of rotation}}=-\frac{\tau\theta}{I}$$

where I is the moment of inertia of the suspending system about the axis of rotation.

The angular acceleration is thus proportional to angular displacement and is directed oppositely to it. The motion is, therefore, simple harmonic. The period of oscillation is given by

$$T=2\pi\sqrt{\frac{I}{\tau}}$$

To determine the moment of inertia of a body, the following observations are to be made :

(*a*) Suspend a V-shaped cradle A (fig. 40) by a wire which is fixed at its upper end to a torsion head B in a cage with glass walls and a glass top. It is then set to torsional oscillations† by twisting the wire a little by turning the suspension head. The period of oscillation is determined by noting the time taken to execute some 50 oscillations by a stop-watch in the usual way. A beam of light may conveniently be reflected from a mirror M fixed to the cradle on a translucent screen and the time of oscillation determined from observations made on the reflected light. Let T be the period of oscillation and I the moment of inertia of the suspending system.

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* See Chapter IX.

†Care must be taken that the rod is not set to pendulum oscillations. If the cradle is set to oscillations like a pendulum these may be stopped by lightly touching the suspension wire just above cradle by the finger.

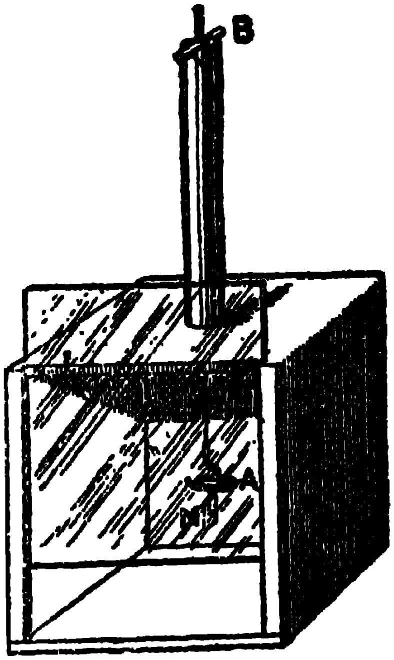

Fig. 40.

(*b*) Place a body the moment of inertia of which can be calculated, such as a rectangular bar, on the cradle so that its centre of mass lies on the prolongation of the suspending wire. The system is then set to oscillations as before. Let T_1 be the period of oscillation and I_1, the moment of inertia of the bar about the line of vibration.

(*c*) The bar is removed and the body of which the moment of inertia is to be determined about an axis is then mounted on the same cradle so that the suspending wire represents the axis about which the moment of inertia is to be determined. The system is set to torsional oscillations and the period of oscillation is found out as before. Let the period be T_2 and I_2 the required moment of inertia of the body. Thus we have

$$T=2\pi\sqrt{\frac{I}{\tau}} \quad \text{Or,} \quad I=\frac{\tau}{4\pi^2}T^2 \quad \ldots \quad (1)$$

$$T_1=2\pi\sqrt{\frac{I+I_1}{\tau}} \quad \text{Or,} \quad I+I_1=\frac{\tau}{4\pi^2}T_1^2 \quad \ldots \quad (2)$$

Also

$$T_2=2\pi\sqrt{\frac{I+I_2}{\tau}} \quad \text{Or,} \quad I+I_2=\frac{\tau}{4\pi^2}T_2^2 \quad \ldots \quad (3)$$

From (1) and (2) $I_1=\frac{\tau}{4\pi^2}(T_1^2-T^2)$ ... (4)

From (1) and (3) $I_2=\frac{\tau}{4\pi^2}(T_2^2-T^2)$ ... (5)

Dividing (5) by (4) $\frac{I_2}{I_1}=\frac{T_2^2-T^2}{T_1^2-T^2}$

I_1 being known, I_2 can be calculated from this relation.

Also from (4) $\tau=\frac{4\pi^2 I_1}{T_1^2-T^2}$

This relation admits of the evaluation of the *torsional rigidity* of the wire.

In the *C. G. S.* system of measurement the moment of inertia will be expressed in gm.-(cm)^2 and in the *F. P. S.* system in lb. -(foot)^2.

(2) **Fly Wheel**—A fly wheel (Fig 41)* consists of a heavy wheel capable of rotation about a horizontal axle; the axle being supported on ball bearings to reduce friction as much as possible.

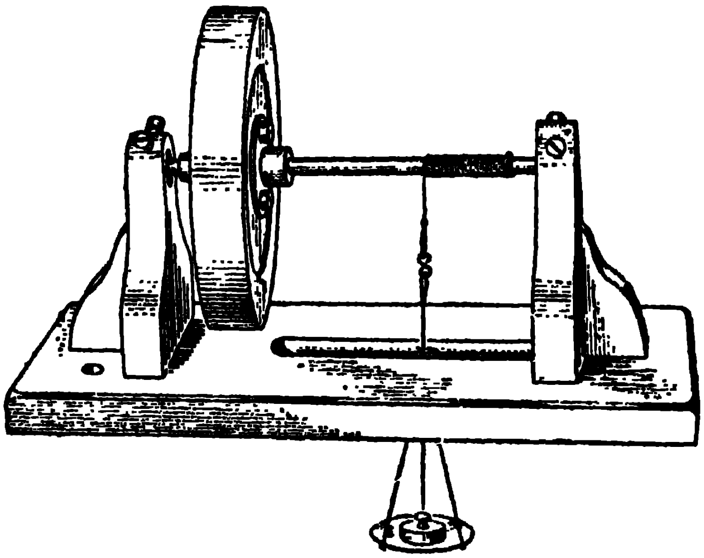

Fig. 41

To determine its moment of inertia about its axis of rotation a cord is wrapped at one of its ends to the axle and is turned round the axle several times, the other end of the cord carries a scale-pan on which a mass is placed. The length of the cord is so adjusted that when the mass reaches the floor in its descent the other end of the cord just leaves the axle. The mass m (including that of scale-pan) is allowed to fall to the floor through a known height x. The loss of *P.E.* of the mass$=mgx$.

---

* The figure is reproduced from Smith's General Properties of Matter.

Let I be the moment of inertia of the fly wheel about the axis of rotation, ω, its angular velocity when the mass strikes the floor and a be the radius of the axle.

Gain of $K. E.$ of the fly wheel$=\frac{1}{2}I\omega^2$

Gain of $K. E.$ of the mass$=\frac{1}{2}ma^2\omega^2$

Equating the loss of $P. E.$ of the mass to the gain of $K. E.$ of the fly wheel and of the mass

$$mgx=\frac{1}{2}I\omega^2+\frac{1}{2}ma^2\omega^2$$

$$\therefore \quad I=\frac{2mgx-ma^2\omega^2}{\omega^2} \quad \ldots \quad \ldots \quad (1)$$

To determine ω at the instant when the cord leaves the axle, a mark is made on the circumference of the wheel which can be seen when the cord leaves the axle. The wheel is then allowed to continue in its rotation (without the mass attached to the axle) until it comes to rest due to friction. If it makes n rotations and t the time required for it, the average angular velocity of the wheel $=2\pi n/t$. This is half the maximum angular velocity ω of the wheel which it possessed when the cord left the axle

$$\therefore \quad \omega=\frac{4\pi n}{t} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Substituting for ω in (1) from (2) the value of I is determined.

(3) **Bifilar Suspension**—The moment of inertia of a body can be determined from the period of oscillation of the body in a horizontal plane under gravity.

Let a heavy uniform rod or bar be suspended by two equal and parallel cords tied to its extremities from a horizontal beam (Fig. 42). Let l be the length of each cord, $2a$ the distance between the cords and m, the mass of the rod or bar. Suppose the bar is made to oscillate in a horizontal plane about a vertical axis through its centre of gravity. If at any time the bar is turned through an angle θ, the displacement of each end of the cord is $a\theta$. Similarly, if the cord becomes inclined to the vertical at an angle ϕ due to the displacement of the bar

$$a\theta=l\phi \quad \ldots \quad (1)$$

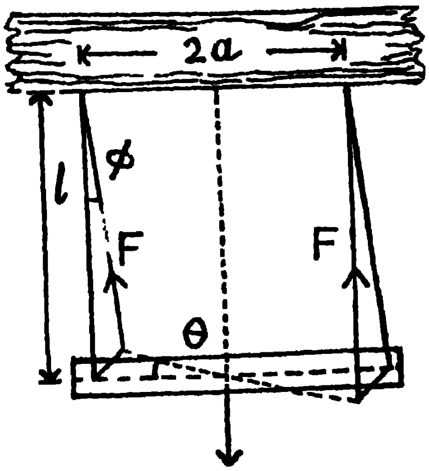

Fig. 42

If F be the tension on either cord at its displaced position, (shown by arrow-heads) the vertical upward component of F is $F\cos\phi$ and horizontal component towards equilibrium position is $F\sin\phi$. The vertical components at the two extremities balance the weight of the bar.

If ϕ is small, $mg=2F\cos\phi=2F$ ... (2)
(as $\cos\phi=1$ nearly).

The horizontal component $F\sin\phi$ at both ends form a couple of moment $2aF\sin\phi=2a\,F\,\phi$ (since ϕ is small)

$$=2aF\,\frac{a\theta}{l}\text{ (from 1)},=\frac{2a^2F}{l}\theta$$

Hence, for oscillatory motion of the rod
$\frac{d^2\theta}{dt^2}=\frac{2a^2F\theta}{lI}$ where I is the moment of inertia of the rod about a vertical axis through $C.G.$ of the rod. Also $\frac{d^2\theta}{dt^2}$ and θ are oppositely directed. Hence, the motion is simple harmonic. The period is given by

$$T=2\pi\sqrt{\frac{lI}{2a^2F}}=2\pi\sqrt{\frac{lI}{mga^2}}\text{ from (2)}$$

$$\therefore\quad I=\frac{mga^2T^2}{4\pi^2l}$$

This method may also be applied to find g if the value of I is known.

Examples

1. Show that the kinetic energy of a thin rod of length l and mass m per unit length rotating about an axis through the middle point and perpendicular to its length with an angular velocity ω is $\frac{1}{24}m\,\omega^2l^3$. (C. U. 1948)

$K.E.$ of rotation$=\frac{1}{2}I\omega^2$ ($I=M.\ I.$ about the axis of rotation)

For a thin rod $I=\frac{1}{12}Ml^2$ where M=mass of the rod

$$=\frac{1}{12}ml^3\text{ (as }M=ml)$$

$$\therefore\quad K.\ E.=\frac{1}{2}.\ \frac{1}{12}ml^3\omega^2=\frac{1}{24}m\omega^2l^3$$

2. A circular disc of mass m and radius r is set rolling on a table. If ω be the angular velocity show that the total energy E is given by

$$E=\tfrac{3}{4}\,m\,r^2\omega^2 \qquad \text{(P. U. \& E. P. U. 1950)}$$

The kinetic energy of the disc is partly translatory and partly rotational.

The translatory $K.\ E.=\frac{1}{2}mv^2=\frac{1}{2}mr^2\omega^2$ (Art. 33)

The rotational $K.\ E.=\frac{1}{2}\,I\,\omega^2=\frac{1}{2}.\frac{1}{2}mr^2.\ \omega^2=\frac{1}{4}mr^2\omega^2$

as I the moment of inertia about the axis of rotation

$$=\tfrac{1}{2}mr^2$$

∴ the total $K.\ E.=\frac{1}{2}mr^2\omega^2+\frac{1}{4}mr^2\omega^2=\frac{3}{4}mr^2\omega^2$.

3. A hoop 3 ft. in diameter weighs 2 lb. Find the kinetic energy of the hoop when it is rolling along a horizontal road at at a linear speed of 7 miles per hour.

7 miles per hour$=\dfrac{7\times1760\times3}{60\times60}$ or $\dfrac{154}{15}$ ft. per sec.

∴ The hoop makes $\dfrac{154}{15\times\pi\times3}$ rotations per sec,

∴ Its angular velocity$=\dfrac{154\times2\pi}{15\times\pi\times3}$ or, $\dfrac{308}{45}$ radians per sec.

Its translatory $K.\ E.=\frac{1}{2}\times2\times\left(\dfrac{154}{15}\right)^2$ or 105·4 ft. poundals

The moment of inertia of the hoop about its axis of rotation

$=Mr^2$ (Art. 73 (5) cor. ii)$=2\times\left(\frac{3}{2}\right)^2$ or $\frac{9}{2}$ lb ft^2.

Its rotational $K.\ E.=\frac{1}{2}\times\frac{9}{2}\times\left(\dfrac{308}{45}\right)^2$ or 105·4 ft. poundals

∴ its total $K.\ E.=210·8$ ft. poundals or 6·59 ft. lbs.

4. A fly wheel is in the form of a uniform circular disc; its radius is 2 ft., and mass 2 lbs. Find the work which must be done on the fly wheel to increase its speed of revolution from 10 to 20 revolutions per second. [Madras B.A. 1947)

Moment of inertia of the fly wheel about its axle $(I)=\frac{1}{2}mr^2$

$$=\tfrac{1}{2}\times2\times4=4\text{ lb ft}^2$$

Its energy at the first stage$=\frac{1}{2}I\omega_1^2=\frac{1}{2}\times 4(2\pi\times 10)^2$ ft. poundals

Its energy at the second " $=\frac{1}{2}I\omega_2^2=\frac{1}{2}\times 4(2\pi\times 20)^2$ft. poundals

Increase in energy$=8\pi^2\{(20)^2-(10)^2\}=8\times 300\pi^2$
$=2\cdot366\times 10^4$ ft. poundals.

Hence, the work done on the fly wheel$=2\cdot366\times 10^4$ ft. poundals.

Exercise

1. What is meant by the moment of inertia of a body about an axis? What is its physical significance?

Calculate the moment of inertia of a thin circular ring of mass M and radius a about an axis in the plane of the ring (a) when the axis passes through the centre, and (b) when it is at a distance r from the centre. (*C.* U. *1936*)

2. Find an expression for the moment of inertia of a thin rod about an axis passing through its end and is at right angles to the length of the rod.

A rod weighing 10 lbs. and of length 9 feet revolves 50 times in a minute; find its kinetic energy. Ans. 8697·5 ft. poundals. (*C.* U. *1940*)

3. Prove that the period of torsional vibration of a rod suspended by a wire is given by

$$T=2\pi\sqrt{\frac{I}{\tau}}$$

where I=moment of inertia and τ=restoring torque.

How can this method be used for finding the moment of inertia of any body?

4. Explain moment of inertia, radius of gyration and moment of momentum and write down their dimensions and the units in which they are measured.

Given the moment of inertia of a body about an axis through its centre of gravity, determine its value about any other parallel axis. (*C.* U. *1944*)

5. Explain what you mean by the moment of inertia and the radius of gyration of a body about an axis. Calculate the moment of inertia of a circular cylinder about its own axis. What is the radius of gyration about that axis?

Describe briefly a method of experimentally obtaining the value of moment of inertia of cylinder. (*C.* U. *1946*)

6. Explain what you mean by the moment of inertia of a body about an axis.

Calculate the moment of inertia of a thin circular ring of a mass M and radius a about an axis in the plane of the ring,

(i) when the axis passes through the centre.

(ii) when it is at a distance from the centre. (*C.* U. *1951*)

7. State and prove the theorem of parallel axes for moment of inertia and deduce an expression for the moment of inertia of a bar of square cross-section about an axis perpendicular to the length of the bar and passing through one of the corners. (*A.* U. *1951*, *P.* U. & *E. P.* U. *1952*)

8. Explain what is understood by the term Moment of Inertia of a body. A couple of 10^8 dyne cms. is applied to a fly wheel of mass 10 kgs. and radius of gyration 50 cms. What is the resulting angular acceleration ? Give the proof of the formula used in the solution of the above problem. (*R.* U. *1952*)

[Ang. accn. $= \dfrac{\text{torque}}{\text{moment of inertia}} = \dfrac{10^8}{MK^2} = \dfrac{10^8}{10{,}000 \times (50)^2}$

$=4$ radians per sec. per sec.]

9. Define the moment of inertia and explain its physical significance. Calculate the moment of inertia of a cylinder of uniform density about an axis passing at right angles to the axis of the cylinder through its middle point. (*L.* U. *1949*)

10. Find the moment of inertia of a solid homogeneous sphere about any diameter. Investigate its motion when projected up an inclined plane with an initial velocity. (*R.* U. *1950*)

11. Define the terms (a) moment of inertia and (b) radius of gyration. Derive expressions for the moment of inertia of a circular disc about an axis passing through its centre and perpendicular to its plane. Indicate an experimental method of determining the moment of inertia. (*N.* U. *1952*)

12. Show that in rotational motion the moment of inertia plays the same role as mass does in linear motion. Derive an expression for the moment of inertia of a sphere about its diameter. (P. U. & *E. P.* U. *1953*)

13. Find the moment of inertia of a hollow sphere about a diameter. (*N.* U. *1954*)

14. Define moment of inertia. Determine the moment of inertia of a disc about a perpendicular axis passing through its centre, and then deduce its moment of inertia about a diameter. (*A.* U. *1946*)

CHAPTER VII

THE ACCELERATION OF GRAVITY AND THE PENDULUM

76. The Acceleration of Gravity—The acceleration of gravity is the acceleration produced in any body moving under the earth's attractive force. It is sometimes called the *intensity of gravity* as it is the measure of the weight of unit mass (Chapter VIII) This acceleration of gravity (represented by g) varies from place to place on the surface of the earth due to two causes—(1) Due to the spheroidal shape of the earth, the equatorial diameter is larger than the polar diameter and consequently the acceleration of gravity in the equatorial region is · less than that in the polar region. (2) Due to the rotation of the earth round its axis, bodies on different latitudes are moving with different speed. The speed of a body on the equator is greater than the speed of a body near the polar region. This causes an increase of the centrifugal acceleration near the equator over that near the polar region. The acceleration of gravity, actually measured at a particular place, is the acceleration due to the earth's attraction minus the centrifugal acceleration of the earth's rotation.

77. Variation of g with latitude : Effect of Earth's rotation on the value of g—

Let $ESQN$ represent a section of the earth assumed spherical of radius a with O as centre (fig. 43), EOQ the equator and NS the axis of rotation. P is a point of latitude λ. If the earth were stationary, a body at P would have been attracted along PO.

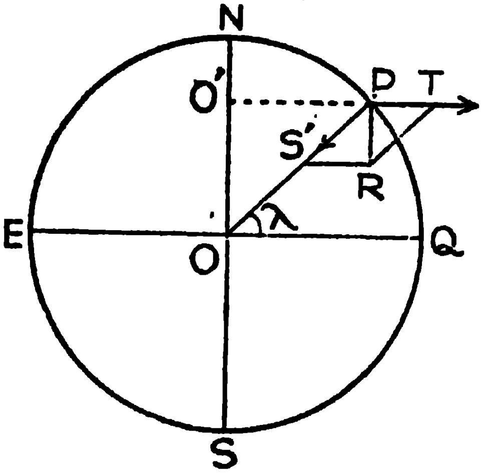

Fig. 43.

Let $\bar{g}$ be acceleration under this condition represented by PS'. Due to the rotation of the earth round its axis P describes a circle of radius $PO'=r$ where PO' is perpendicular from P on the axis of rotation NS. Evidently $r=a\cos\lambda$. The centrifugal acceleration of the body $=\omega^2 a\cos\lambda$ (Art. 33) represented by PT. The resultant acceleration g_λ obtained by compounding PT and PS' is along PR and is of magnitude given by

$$g_\lambda = \sqrt{PS'^2 + PT^2 - 2PS'PT\cos\lambda}$$

$$= \sqrt{PS'^2 - 2PS'PT\cos\lambda}$$

(as PS' is large compared to PT)

$$= PS'\left(1 - \frac{2PT}{PS'}\cos\lambda\right)^{\frac{1}{2}} = PS' - PT\cos\lambda$$

$$= g - \omega^2 a\cos^2\lambda \text{ (as } PT = \omega^2 a\cos\lambda) \quad \ldots \quad (1)$$

Corresponding to the poles $\lambda = 90^\circ$ and $g_{90} = g$

$$\therefore \quad g_\lambda = g_{90} - \omega^2 a\cos^2\lambda \quad \ldots \quad (2) \text{ from } (1)$$

Taking $a = 6{\cdot}37\times10^8$ cm. and $\omega = \dfrac{2\pi}{86400} = 7{\cdot}27\times10^{-5}$ radians/sec.

$\omega^2 a = 3{\cdot}37$ cm./sec.^{2}

$$g_{45} = g_{90} - \tfrac{1}{2}\omega^2 a \quad \ldots \quad \ldots \quad \text{from } (2)$$

$$= g_{90} - 1{\cdot}69$$

$$\therefore \quad g_{90} = g_{45} + 1{\cdot}69 = 980{\cdot}62 + 1{\cdot}69 = 982{\cdot}31 \text{ cm./sec.}^2 \ \ldots \ (3)$$

(the accepted value for g_{45} being $980{\cdot}62$)

$$\therefore \quad g_\lambda = 982{\cdot}31 - 3{\cdot}37\cos^2\lambda \quad \ldots \quad \text{from (2) and (3)}$$

A better formula is $g_\lambda = 983{\cdot}21 - 5{\cdot}17\cos^2\lambda$

The difference is due to the fact that the earth is not a perfect sphere and that it is not homogeneous.

78. Variation of g with Altitude—Let the earth be considered to be a homogeneous sphere or to consist of concentric spherical shells of uniform volume density. Following Chapter VIII the intensity of gravity or acceleration due to gravity at any point outside the earth's surface will vary inversely as the square of its distance from the centre of the earth. Let g and g' be the values of the acceleration of gravity at the earth's surface and at a height h above the surface of the earth respectively. Taking a to be the radius of the earth

$$\frac{g'}{g} = \frac{a^2}{(a+h)^2} = \frac{a^2}{a^2\left(1+\frac{h}{a}\right)^2} = \frac{1}{1+\frac{2h}{a}+\frac{h^2}{a^2}}$$

Neglecting the term $\frac{h^2}{a^2}$ compared to $\frac{2h}{a}$

$$\frac{g'}{g}=\frac{1}{1+\frac{2h}{a}}=\left(1+\frac{2h}{a}\right)^{-1}=1-\frac{2h}{a} \text{ approximately}$$

$$\therefore \quad g'=g\left(1-\frac{2h}{a}\right)$$

This assumes that there is no elevated mass, such as a mountain, between the earth's surface and the point at the height h above it.

79. Measurement of the Acceleration of Gravity—Among the different methods of measuring the acceleration of gravity, the application of pendulum seems to originate with Galileo. Newton made a large use of the theory of pendulum in his *Principia* and brought home the idea of definiteness of the mass of a body. Since then the subject has been studied notably by Bouguer, Kater, Bessel, Repsold and others.

80. The Simple Pendulum—The simple pendulum consists of 'a heavy particle of mass suspended by a weightless inextensible string from a rigid support. The distance between the point of suspension and the particle of mass is called the length of the pendulum. A rigorous simple pendulum is impossible to realise. A small metal sphere suspended by a long thin thread is a near approach to a simple pendulum. The distance from the point of suspension to the centre of the sphere will then be the length of the simple pendulum.

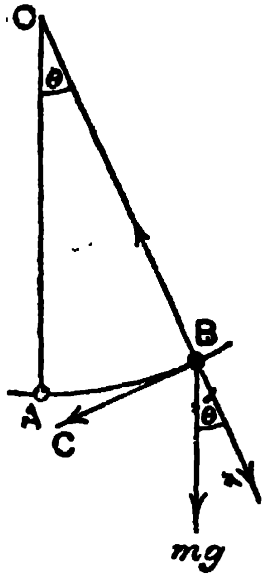

Fig. 44

Let OA (fig. 44) represent a simple pendulum of length l in its undisturbed position. OB represents its position when displaced through an angle θ. The force on the bob at B is its weight mg, acting vertically downwards. This force may be resolved into two components—

(1) $mg \cos\theta$ along OB (produced) and

(2) $mg \sin\theta$ along BC, the tangent at B to the circular path on which the bob moves.

Out of these, the component (2) alone provides the restoring force on the bob ; the component (1) simply produces tension on the string along BO. The component (2) produces an acceleration of the bob along the tangent BC which

$$= g \sin\theta$$

$$= g\,\theta, \text{ if } \theta \text{ is small}$$

$$= g\frac{\text{arc } AB}{l} \text{(where } l = \text{the length of the pendulum)}$$

Again, since θ is small, the arc AB will be equal to the chord AB, which measures the displacement of the bob from its undisturbed position.

Hence, the acceleration of the bob is directed to the fixed point A in its path and is equal to $\frac{g}{l}$ times its displacement from that point.

The motion of the bob is thus simple harmonic and its period of oscillation is given by

$$T = 2\pi\sqrt{\frac{l}{g}} \quad \dots \quad \dots \quad \text{(Art. 47)}$$

Alternative Method—It is sometimes useful to investigate the motion of the simple pendulum from the energy equation.

Let OA represent the undisturbed position of the pendulum (fig. 45). It is displaced to a position OB making an angle α with OA and then released. Therefore α measures the angular amplitude. Let after a time t, the pendulum, move through the position OC, at which its angular displacement is θ. If BB' and CC' are the perpendiculars drawn upon OA, and l, the length of the pendulum, $OB' = l\cos\alpha$ and $OC' = l\cos\theta$.

In moving from B to C, the bob of the pendulum has fallen through the vertical height $B'C'$. If m be the mass of the bob, its loss of potential energy, in falling through the vertical height $B'C' = mg\,B'C' = mg(OC' - OB')$

$$= mgl\,(\cos\theta - \cos\alpha) \quad \dots \quad (1)$$

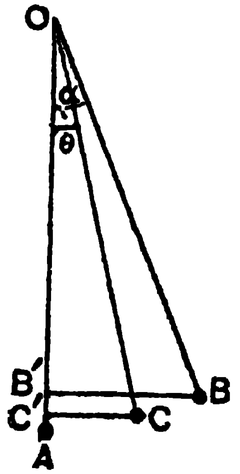

Fig. 45

This loss of potential energy of the pendulum appears as its kinetic energy in the position OC.

The angular velocity of the pendulum in the position OC, which is measured by the rate of angular displacement

$$= \frac{d\theta}{dt} \text{ and its linear velocity} = l\frac{d\theta}{dt}$$

$$\text{its kinetic energy} = \tfrac{1}{2}ml^2\left(\frac{d\theta}{dt}\right)^2 \quad \dots \quad (2)$$

Hence, equating (2) to (1)

$$\frac{1}{2}ml^2\left(\frac{d\theta}{dt}\right)^2=mgl\,(\cos\theta-\cos a)$$

$$\text{or,}\quad\left(\frac{d\theta}{dt}\right)^2=\frac{2g}{l}\,(\cos\theta-\cos a)$$

Differentiating with regard to t and remembering that a is a constant

$$2\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2}=-\frac{2g}{l}\sin\theta\,\frac{d\theta}{dt}$$

$$\therefore\quad\frac{d^2\theta}{dt^2}=-\frac{g}{l}\sin\theta$$

$$\text{or,}\quad\frac{d^2\theta}{dt^2}=-\frac{g}{l}\theta\ldots\ldots\ldots\ldots\ldots\ldots(3)\ (\text{if }\theta\text{ is small})$$

$\frac{d^2\theta}{dt^2}$ measures the angular acceleration. Equation (3) shows that the angular acceleration is proportional to angular displacement and the negative sign indicates that the angular acceleration is directed oppositely to the angular displacement. The motion of the bob is thus simple harmonic. The time of oscillation is given by

$$T=2\pi\sqrt{\frac{l}{g}}\qquad\ldots\qquad\ldots\qquad(\text{Art. }47)$$

81. The period of oscillation of a simple pendulum deduced in the last article, strictly holds good for an ideal simple pendulum provided the arc of swinging of the bob is infinitely small. In 1747 Bernouilli showed that for an arc of small finite length, the observed period of oscillation is given by

$$T_0=2\pi\sqrt{\frac{l}{g}}\left(1+\frac{a^2}{16}\right)$$

where a is the angular amplitude in radian. For decrease in amplitude occurring during observation. the expression takes the form

$$T_0=2\pi\sqrt{\frac{l}{g}}\left(1+\frac{a_1a_2}{16}\right)$$

where a_1 and a_2 are the initial and final amplitudes during the time of observation.

Hence, the period of oscillation corrected for finite arc of swinging is given by

$$T=2\pi\sqrt{\frac{l}{g}}=T_0\Big/\left(1+\frac{a_1a_2}{16}\right)=T_0\left(1-\frac{a_1a_2}{16}\right)$$

82. The Conical Pendulum—The conical pendulum consists of a heavy particle B suspended by a light string AB (fig. 46), the end A of which is fixed.

The particle describes a circular path uniformly on a horizontal plane. The string AB in revolving, generates a conical surface.

Let O be the centre of the circular path of the particle having radius r; l the length of the pendulum, m the mass of the particle and v its uniform speed round the path.

The forces acting on the particle at B are—

(1) The tension P of the string in the direction BA and

(2) The weight of the particle mg acting vertically downwards.

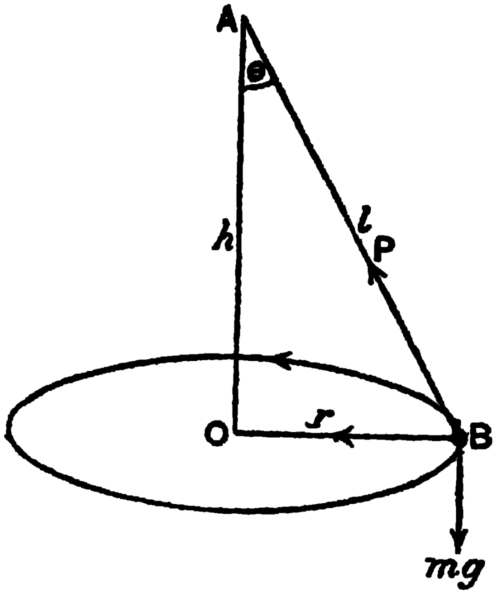

Fig. 46

The resultant of these two forces must form a single force along BO and is of magnitude $\frac{mv^2}{r}$ which is the centripetal force on the particle necessary for its circular motion.

If θ be the angle between AO and AB, the component of the tension P at B along $BO = P \sin \theta$.

As the weight of the particle has no component along BO, we have

$$P \sin \theta = \frac{mv^2}{r} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

The other component of P at B acting vertically upwards $= P \cos \theta$. This balances the weight of the particle.

$$\therefore \; P \cos \theta = mg \quad \ldots \quad \ldots \quad . \quad (2)$$

Dividing (1) by (2) we have

$$\tan \theta = \frac{v^2}{gr}$$

Again, $\tan \theta = \frac{OB}{AO} = \frac{r}{h}$ (where $AO = h =$ height of the conical surface)

$$\therefore \; \frac{r}{h} = \frac{v^2}{gr} = \frac{r^2\omega^2}{gr}$$

(where $\omega =$ the angular velocity of the particle).

$$\therefore \; \omega = \sqrt{\frac{g}{h}} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

If T is the time of a single revolution of the particle,

$$\omega = \frac{2\pi}{T} \text{ or, } T = \frac{2\pi}{\omega}$$

Hence, from (3) $T = 2\pi\sqrt{\frac{h}{g}} \quad \ldots \quad \ldots \quad \ldots \quad (4)$

If θ, the inclination of the string to the vertical is small, the length of the string $l=h$, and the relation (4) reduces to

$$T=2\pi\sqrt{\frac{l}{g}}$$

The conical pendulum is used as an engine-governor.

From relation (8) we have

$$h=\frac{g}{\omega^2}$$

This shows that the height of the cone is independent of the mass of the particle and of the length of the string AB It depends on the angular velocity ω. If ω_1 and ω_2 are the angular velocities of the particle, when the corresponding heights of the cone are h_1 and h_2,

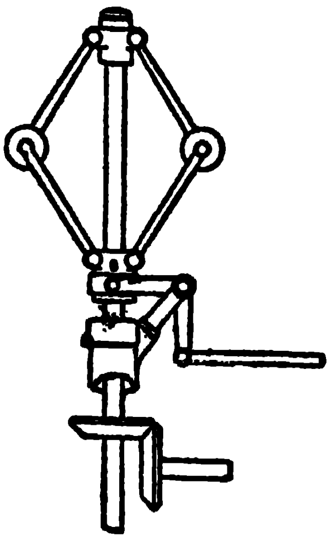

Fig. 47

$h_1=$ and $h_2=$

$$\therefore\ h_1-h_2=g\left(\frac{1}{w_1^2}-\frac{1}{\omega_2^2}\right)$$

Thus if $\omega_2>\omega_1$, $h_2<h_1$, *i.e.*, on increasing the angular velocity, the height of the cone decreases or the particle rises. If $\omega_2<\omega_1$ $h_2>h_1$ *i.e.*, on decreasing the angular velocity, the height of the cone increases or the particle falls. In the engine-governor (fig. 47) there is a vertical spindle driven by the engine. The spindle is provided with two arms near the top which can rise or fall. There are two masses connected to the ends of the two arms. Two other arms connect these masses to a sleeve which can slide up and down the spindle. The movements of the sleeve are communicated to a throttle valve in the steam pipe by means of a bent lever as shown in the diagram. An increase of speed of the engine causes the revolving masses to rise with a consequent rise of the sleeve along the spindle. The rise of the sleeve closes the throttle valve partially, thus reducing the quantity of steam passing into the cylinder of the engine, and hence the speed of the engine drops. A reduction of speed of the engine is followed by an opposite action and more steam enters the cylinder of the engine and increases the speed of the engine thereby.

83. The Compound Pendulum—If it were possible to obtain an ideal simple pendulum, *i.e.*, a particle of mass suspended from a rigid support by a weightless inextensible string, the acceleration of gravity at any place could have been found from its period of oscillation and its length.

Although such a pendulum is not realisable, still it is possible to calculate what should be the length of a simple pendulum which vibrates in the same period as a body of an arbitrary shape, free to vibrate about a horizontal axis. Such a rigid body of arbitrary shape and size which can vibrate about a horizontal axis is known as a *compound pendulum*. An ideal simple pendulum, the period of which is the same as that of the body, is known as its **equivalent simple pendulum.**

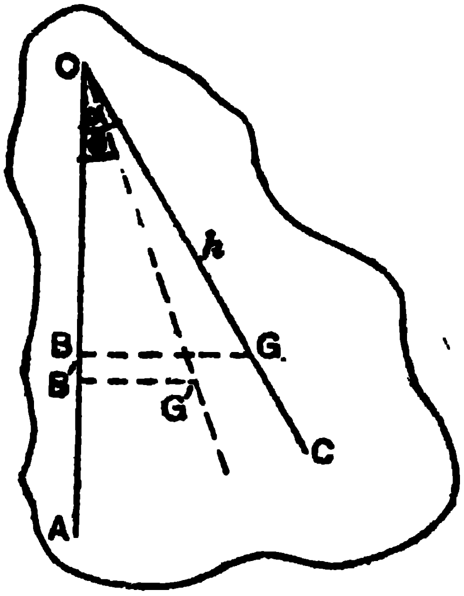

Fig. 48

Consider a body capable of vibration about a horizontal axis perpendicular to the plane of the paper passing through O. Fig. 48 shows a sectional diagram of the body by a vertical plane perpendicular to the axis, and passing through the centre of gravity G of the body. In the undisturbed position of the body, the centre of gravity, at which the weight Mg of the body may be supposed to act, remains vertically below O on the line OA. Let the body be initially displaced such that the line OG makes an angle α with its undisplaced position OA and subsequently released. After a time t, suppose the line OG passes through the position OG', being inclined at θ to OA. If GB and $G'B'$ are the normals on OA, the centre of gravity falls through the vertical height BB' which is equal to $OB'-OB=h(\cos\theta-\cos\alpha)$, where h = the distance OG or OG'. The loss of potential energy of the body in passing from initial position to its position after the time t

$$=Mgh\,(\cos\theta-\cos\alpha) \quad \ldots \quad \ldots \quad (1)^*$$

This loss of potential energy of the body must be equal to its gain in kinetic energy.

If Mk^2 represents the moment of inertia of the body, about an axis perpendicular to the plane of the paper through its centre of gravity, the moment of inertia of the body about its axis of rotation (by the theorem of parallel axes)

$$=M(h^2+k^2)\dagger \quad \ldots \quad \ldots \quad \ldots \quad \text{(Art. 72)}$$

---

[*] Compare Art. 80.

[†] The centre of gravity and the centre of mass are practically two coincident points. The theorem which is proved for laminar bodies also holds for three-dimensional bodies.

and the rotational kinetic energy of the body in its position after the time t

$$=\tfrac{1}{2}M(h^2+k^2)\omega^2 \qquad \ldots \qquad \ldots \qquad (2)\ (\text{Art. } 64)$$

where ω is the angular velocity of the centre of gravity in the position G'. Equating (1) and (2)

$$\tfrac{1}{2}M(h^2+k^2)\omega^2=Mgh\,(\cos\theta-\cos\alpha)$$

Cancelling like factors and putting $\omega=\dfrac{d\theta}{dt}$

$$\tfrac{1}{2}(h^2+k^2)\left(\frac{d\theta}{dt}\right)^2=gh\,(\cos\theta-\cos\alpha)$$

$$\text{or,}\quad \left(\frac{d\theta}{dt}\right)^2=\frac{2gh}{h^2+k^2}(\cos\theta-\cos\alpha)$$

Differentiating both sides with regard to t and remembering that α is a constant,

$$2\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2}=-\frac{2gh}{h^2+k^2}\sin\theta\frac{d\theta}{dt}$$

Cancelling like factors and putting $\sin\theta=\theta$, when θ is small.

$$\frac{d^2\theta}{dt^2}=-\frac{gh}{h^2+k^2}\theta^{*}$$

The abve expression for angular acceleration shows that angular acceleration is proportional to the angular displacement and is directed oppositely to it.

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[*] The result can also be deduced from the analogy between rectilinear and rotational motion (Art 68). The weight of the pendulum Mg may be supposed to act vertically downwards through G' its centre of gravity in the displaced position OG'. This force with its reaction at O acting vertically upwards constitutes a couple of moment $Mgh\sin\theta$ or $Mgh\theta$, since θ is small. This is the moment of the turning couple on the pendulum. The moment of inertia of the pendulum about the axis of rotation through O is $M(h^2+k^2)$.

$$\text{From Art. 68,}\quad \frac{d^2\theta}{dt^2}=-\frac{Mgh\theta}{M(h^2+k^2)}=-\frac{gh\theta}{h^2+k^2}$$

the negative sign is due to the fact that the angular acceleration and displacement are oppositely directed.

The motion of the pendulum is, therefore, simple harmonic. Its period of oscillation is given by

$$T=2\pi\sqrt{\frac{h^2+k^2}{gh}} \quad \ldots \quad \ldots \quad (3)$$

Comparing the expression (3) with the period of oscillation of an ideal simple pendulum, *viz.* $T=2\pi\sqrt{\frac{l}{g}}$, we see that the length of an ideal simple pendulum having the same period of oscillation as that of the body, will be given by

$$l=\frac{h^2+k^2}{h} \quad \ldots \quad \ldots \quad (4)$$

Such an ideal simple pendulum of which the length is given by (4) is the *equivalent simple pendulum.*

84. Centres of Suspension and Oscillation and their Interchangeability—

The length of the equivalent simple pendulum found in equation (4) of the last article is given by

$$l=\frac{h^2+k^2}{h}=h+\frac{k^2}{h}$$

Hence, if the line OG is produced to C, (fig. 48) such that $GC=\frac{k^2}{h}$ then OC gives the length of the equivalent simple pendulum.

The point O at which the body is suspended is called the *centre of suspension* and the point C is called the *centre of oscillation.*

It can be shown that the centres of suspension and oscillation are interchangeable ; *i.e.*, whether the body is suspended at O or at C, its time of oscillation will be the same.

When suspended at O, its time of oscillation

$$=2\pi\sqrt{\frac{h^2+k^2}{gh}} \quad (1), \text{ where } h=OG.$$

Similarly, when suspended at C its time of oscillation.

$$=2\pi\sqrt{\frac{CG^2+k^2}{g\times CG}}=2\pi\sqrt{\frac{CG+\frac{k^2}{CG}}{g}}$$

$$=2\pi\sqrt{\frac{\frac{k^2}{h}+h}{g}}\left(\text{since } CG=\frac{k^2}{h}\right)$$

$$=2\pi\sqrt{\frac{k^2+h^2}{gh}} \qquad \dots \qquad (2)$$

The expressions (1) and (2) are identical, showing that the centres of suspension and oscillation are interchangeable. The distance between these two points is evidently the length of the equivalent simple pendulum.

85. Other Properties of Centres of Suspension and Oscillation—

The length of the equivalent simple pendulum is given by

$l=\frac{h^2+k^2}{h}$ (Equation 4, Art. 83)

Since $l=OC$ and $h=OG$, we have

$$OC=\frac{OG^2+k^2}{OG} \quad \text{or, } OC.\ OG=OG^2+k^2$$

$$\text{or, } OG\ (OC-OG)=k^2 \quad \text{or, } OG.GC=k^2$$

In other words, the product of the distances of the centre of suspension and the centre of oscillation from the centre of gravity of the pendulum is equal to the square of the radius of gyration of the pendulum about an axis through the centre of gravity perpendicular to the line joining the centres of oscillation and suspension.

Again from the equation (3) of Art 83.

$$T^2=\frac{4\pi^2(h^2+k^2)}{gh}$$

$$\text{or, } h^2-\frac{T^2g}{4\pi^2}h+k^2=0$$

This quadratic equation in h has two roots the sum and product of which are both positive. Hence both the roots are positive. Thus, for a given value of T there are two positive values of h in general; *i.e.*, there are two positions of O, the centre of suspension on the same side of G.

Similar reasoning shows that there are two points on the OG produced situated on the side opposite to O from G about which the period will have the same value T. *Therefore, on any chosen line* through G there are four points about which the period of oscillation of the pendulum is the same, the axis of rotation being always perpendicular to the chosen line. The distance between any pair of such points situated on opposite sides of G and at unequal distances from it, is the length of the equivalent simple pendulum.

86. Minimum Period of Oscillation.—If a graph is drawn with the periodic time (t) as ordinate and distances of the centres of suspension and oscillation on the two sides respectively from the centre of gravity as abscissæ, it will be as shown in fig 49.

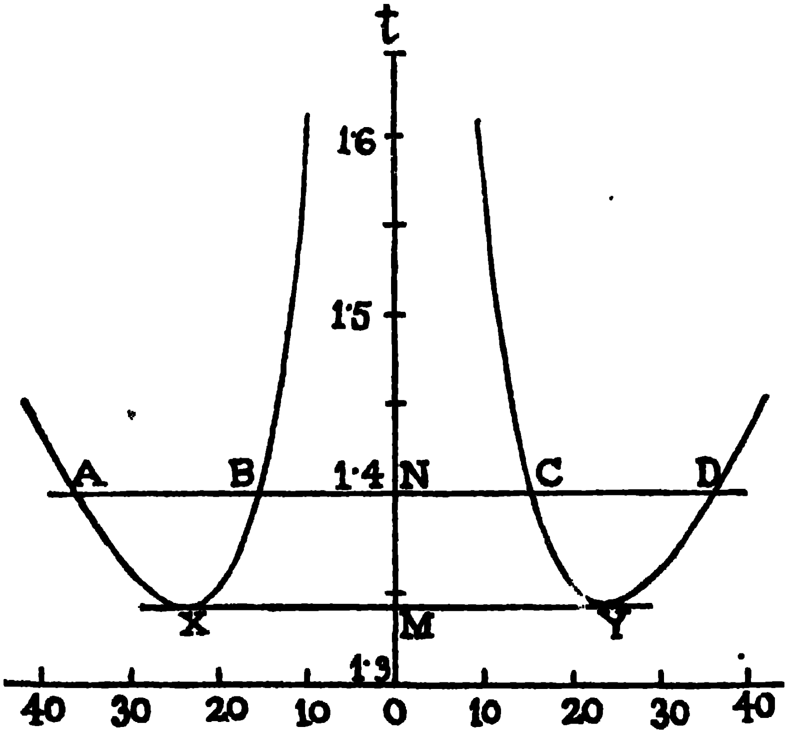

Fig. 49

If a line $ABCD$ parallel to abscissa is drawn cutting the curve at the four points A, B, C and D, the period represented by

ON corresponding to these four points will be the same and the distance *AC* or *BD* is the length of the equivalent simple pendulum.

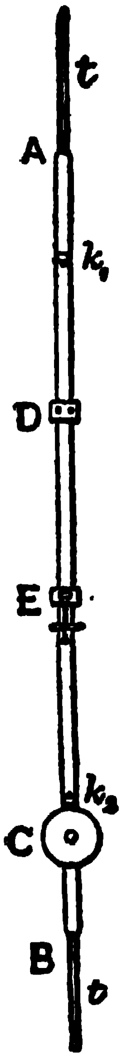

Fig. 50

If a straight line *XY* is drawn parallel to the abscissa tangential to the curve meeting it at *X* and *Y*, the period represented by *OM* corresponding to these points situated at equal distances from *M* is the minimum. The minimum period can be calculated as follows—

Differentiating the equation $T^2 = \frac{4\pi^2(h^2+k^2)}{gh}$ ...(1) of Art. 85 with regard to h

$$2T\frac{dT}{dh} = \frac{4\pi^2}{g} \cdot \frac{h^2-k^2}{h^2}$$

This equation determines the variation of T with h.

Putting $\frac{dT}{dh} = 0$, T is a minimum when $h = k$

∴ minimum value of the period is given by (putting $h = k$ in (1))

$$T = 2\pi\sqrt{\frac{2k^2}{gk}} = 2\pi\sqrt{\frac{2k}{g}}$$

87. Kater's Pendulum—Captain Kater in 1817 hit upon this idea and for the first time carried it into practice, thus finding the value of g at London with his instrument known as *Kater's convertible pendulum*. The pendulum consists of a metal rod *AB* (fig. 50) at one end of which a heavy lens-shaped mass *C* is fixed. Two steel knife-edges k_1 and k_2 are fixed to this rod with their edges turned towards each other. The distance between the knife-edges is such that the periods of oscillation of the pendulum about both of them are nearly equal. The rod carries two other small masses *D* and *E*. These masses can slide along the rod and can be screwed anywhere on it. Of these the larger mass D is moved about until the times of oscillation about both the knife-edges are almost the same. Then it is screwed in position. The position of the smaller mass *E* is next adjusted to make the times of oscillation about both the knife-edges exactly the same.

Kater determined the times of oscillation about the knife-edges by method of coincidence.[*] He placed a clock behind his convertible pendulum and on the bob of the clock-pendulum a white circle on a black paper was attached. The width of the circle was the same as that of the tail of the convertible pendulum. The convertible pendulum was focussed by a telescope. At the focal plane of the eye-piece of the telescope a slit having width as that of the pendulum-tail *t* was placed. The telescope, the convertible pendulum and the clock were so adjusted that in the undisturbed position of the two pendulms, the white patch of the clock-pendulum covered itself behind the tail of the convertible pendulum and was thus invisible through the slit of the telescope. This is known as the *coincidence*. Both the pendulums were set to vibration and the number of oscillations executed by each during two successive coincidences, when both the pendulums were in the same phase of vibration were counted. From a knowledge of the period of oscillation of the clock-pendulum, the period of the convertible pendulum about any of the knife-edges was thus calculated.

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[*] Horton has described a new method on coincidence, Phil. Trans., A. 204, (1904).

The distance between the knife-edges is then measured, which gives the length of the equivalent simple pendulum. From this the value of g is calculated.

88.. **Bessel's Modification**—The adjustment of the masses for exact equality of periods about the two knife-edges is extremely tedious. Bessel, in 1826, showed that an exact equality of the time-periods about the two knife-edges was not absolutely necessary, if the two time-periods were nearly equal and the centre of gravity of the pendulum were near to one of the knife-edges.

Let T_1 and T_2 be the periods of oscillation about the two knife-edges and let h_1 and h_2 be the distances of the centre of gravity of the pendulum from these knife-edges respectively.

$$\therefore \quad T_1 = 2\pi\sqrt{\frac{h_1^2+k^2}{gh_1}}$$

$$\text{and } T_2 = 2\pi\sqrt{\frac{h_2^2+k^2}{gh_2}} \quad \ldots \text{ from (3), Art. 83}$$

$$\therefore \quad \frac{gh_1T_1^2}{4\pi^2} = h_1^2+k^2 \text{ and } \frac{gh_2T_2^2}{4\pi^2} = h_2^2+k^2.$$

Subtracting,

$$\frac{g}{4\pi^2}(h_1T_1^2 - h_2T_2^2) = h_1^2 - h_2^2$$

$$\therefore \quad \frac{4\pi^2}{g} = \frac{h_1T_1^2 - h_2T_2^2}{h_1^2 - h_2^2} = \frac{T_1^2+T_2^2}{2(h_1+h_2)} + \frac{T_1^2-T_2^2}{2(h_1-h_2)}$$

h_1+h_2 measures the distance between the knife-edges which is determined accurately, and finding T_1 and T_2 by the method of coincidence, the first term of the above expression is accurately found out. But h_1 and h_2 and hence h_1-h_2 cannot be determined accurately. To measure h_1 and h_2 the pendulum is balanced in a horizontal position on a knife-edge. The distances of the knife-edges of the pendulum from the balancing knife-edge give h_1 and h_2 whence $h_1 - h_2$ is found out. This rough way of finding $h_1 - h_2$ does not alter the value of the expression for g appreciably ; because T_1 and T_2 are nearly the same, and the centre of gravity of the pendulum being near to one of the knife-edges, $h_1 - h_2$ is large. The term $\frac{T_1^2 - T_2^2}{2(h_1 - h_2)}$ of the expression is thus of very small magnitude compared to the term $\frac{T_1^2 + T_2^2}{2(h_1 + h_2)}$ and hence a rough evaluation of the former is sufficient for the determination of g.

89. **Repsold's Pendulum**—In the above calculation for the time of oscillation, it will be found that the effect of air in which the pendulum oscillates has not been allowed for. Kater considered that the effect of air was simply a buoyancy effect and introduced a correction for this.

Bessel showed* that the effect of air on the pendulum was not only the buoyancy effect but in addition to that, the air due to its motion provided a viscous drag on the pendulum. Bessel showed that both these effects were eliminated out, if the external form of the pendulum were symmetrical about its middle point.

A pendulum. fulfilling these conditions, was constructed by Repsold in 1860. In Repsold's pendulum two rings are fixed at the ends of a bar, Into the rings, are screwed two other short bars having knife-edges. The short bars are provided with two symmetrical bobs, one of which has a large mass compared to the other. The centre of gravity of the pendulum is thus near to one of the knife-edges and the pendulum is symmetrical about its middle point. The periods about both the knife-edges are brought to near equality by screwing the bars in or out.

Examples

1. Find the period of a compound pendulum. A compound pendulum is formed by suspending a heavy ring of radius 490·5 cm. from a string. What is the length of the string when the period is minimum ? Find this period. *(R. U. 1950)*

The moment of inertia of a ring about an axis through its centre and perpendicular to its plane —(an axis parallel to the axis of rotation)$=Mr^2$ where M is its mass and r its radius. Hence, the radius of gyration about the axis$=r$. The period will be a minimum if the distance of its $C.G.$ from the point of suspension is equal to the radius of gyration, or radius r of the hoop. Therefore the length of the string for minimum period$=0$.

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*For a detailed account of Bessel's calculation the student is referred to Poynting and Thomson's Properties of Matter.

The minimum period is given by

$$T=2\pi\sqrt{\frac{2k}{g}}=2\pi\sqrt{\frac{2\times490{\cdot}5}{981}}=2\pi$$
$$=6{\cdot}284 \text{ sec.}$$

2. A uniform square lamina oscillates about a horizontal axis normal to its plane. Find where the axis must pierce the square in order that the period may be a minimum.

The radius of gyration of a square lamina about an axis through its centre perpendicular to its plane

$$=\sqrt{\frac{a^2+a^2}{12}} \text{ or } \frac{a}{\sqrt{6}} \quad \text{Art. 73 (3)}$$

where a is the length of its side. Hence the lamina must be suspended about an axis perpendicular to is plane passing through a point at a distance $\frac{a}{\sqrt{6}}$ from its centre.

3. A uniform circular rod of radius 2 cms. oscillates when suspended from a point on its axis at a distance of 4 cms. from one end. If the length of the rod is one metre, find the point or points from which, if suspended, the periodic time would remain unaltered. (Bombay 1942)

If k be the radius of gyration of the rod about its middle point (C.G.), $k^2=\frac{(100)^2}{12}$ [Art 73(2)]. Another point of suspension will be at a distance $\frac{k^2}{CG}$ (Art. 84)$=\frac{(100)^2}{12\times46}$ or 18·13 cm. from the C.G. on the side of C.G. other than the point of suspension or at a distance (50+18.13) or 68·13 cm. from the same end of the rod.

Two other points of suspension of the same period will be (1) at a distance 18·13 cm. from C.G. on the side of the point of suspension and (2) at a distance (50 – 4) or 46 cms. from C·G. on the other side of the point of suspension. Hence their distances from the same end of the rod are respectively (50 – 18·13) or 31·87 cm. and (50+46) or 96 cms.

Exercise

1. Explain how the value of g at any point on the surface of the earth can be determined with Kater's pendulum. How should the value vary if the experiment be performed at the bottom of a deep mine, assuming the earth to be of uniform density? Does the result agree with actual observation? *[C. U. 1933]*

2. Derive an expression for the periodic time of a compound pendulum. Define centres of suspension and oscillation, and show that they are interchangeable. *[L. U. 1947]*

3. How does g vary with latitude? Derive the necessary expression if the earth were a rotating homogeneous sphere. Describe any instrument you know which is used for measuring variations in g. *[P. U. & E. P. U. 1953]*

4. Give the theory of Kater's pendulum. In what way is it superior to a simple pendulum? *(L. U. 1945)*

5. Write notes on Kater's pendulum. *[B. U. 1954]*

CHAPTER VIII

GRAVITATION AND GRAVITATIONAL CONSTANT

90. Gravitation and Gravity—Between any two bodies, there always exists a mutual force of attraction. This phenomenon of mutual attraction between any two bodies is known as *Gravitation*. When one of the attracting bodies constitutes the earth, the phenomenon is known as *Gravity*. Thus, gravity is the attraction of the earth upon a body, and is only a special case of gravitation.

91. Laws of Gravitation—The laws of gravitation were discovered by Newton. Any particle of mass attracts another at a certain distance apart with a force directly proportional to the product of the masses of the two particles and inversely proportional to the square of the distance between them. Symbolically this may be represented as

$$F=G\frac{mm'}{d^2}$$

where F is the force of attraction between two particles of masses m and m' at a distance d apart. The constant G is known as the *Gravitational constant.* If two particles, each of mass one gramme, are separated by a distance of one centimetre, the force of attraction between them is equal to G. Thus, the gravitational constant (in the C. G. S. System) may be defined as numerically the force of attraction between two particles, each of mass one gramme, separated by a distance of one centimetre.

92. Gravitational Attraction and Potential—In the space surrounding a massive body an attractive force will be exerted on a particle at any point. This attractive force on a *unit mass* placed *at a point* is known as the *intensity of attraction* or *simply attraction at that point.*

The *potential* at a point in the space, is measured by the amount of work done in bringing a unit mass from an infinite distance to that point. It will, however, be noted that the force exerted by the body on the unit mass is always an attractive one. Hence, in bringing the particle of unit mass from infinity to any point at a finite distance from the body, no work is required to be done by the external agency, but work will be done by the gravitational force itself. The work done in carrying the unit mass to a point from infinity is thus represented by a negative quantity which measures the potential at that point due to the massive body. The case is analogous to the electric potential at a point due to a negative charge, or the magnetic potential at a point due to a south pole.

A point of difference between gravitational potential and electric or magnetic potential arises due to the fact that the gravitational attraction between two masses is independent of the intervening medium (Art. 113), while that between two electric charges or magnetic poles depends on the medium between the charges or magnetic poles. Hence, the gravitational potential due to a mass is independent of the medium in which the mass is placed while the electric or magnetic potential due to an electric charge or a magnetic pole is dependent on the medium in which the charged body or magnetic pole is situated.

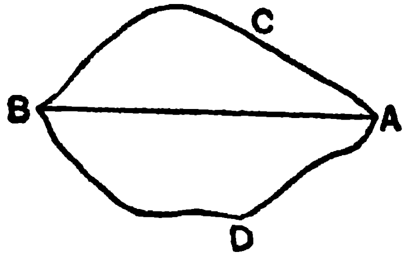

Fig. 51

From the definition of gravitational attraction and potential, a relation can be established between the two. Let V_A and V_B respectively represent the gravitational potentials at two neighbouring points A and B (fig. 51). If F represents the gravitational force of attraction on unit mass acting along the straight line AB, the work done in moving the unit mass from A to $B = F \times AB$. Again, this measures the difference of potential between the two points. Hence,

$$F \times AB = V_B - V_A$$

$$\text{or, } F = \frac{V_B - V_A}{AB}.$$

Thus, the attraction in any direction is measured by the rate of change of potential in that direction with respect to distance.

It can be shown that the work done in moving a unit mass from one point to another is independent of the path traversed by the unit mass.* For, let the unit mass, initially at A (fig. 51), be carried from A to B along the path ACB. The work done by the gravitational force is equal to W_1 (suppose). Subsequently, let it be carried back from B to A along a different path BDA. Let the work done by the external agency be W_2. If the two works W_1 and W_2 are not the same, but $W_1 > W_2$, a net gain of energy is obtained during the cycle of operation, at the end of which the unit mass returns to its previous condition. Similarly, if $W_2 > W_1$, there is a loss of energy during the cycle of operation, although at the end of it the unit mass returns to its previous condition. This is against the law of conservation of energy. Hence, the work done is independent of the path traversed.

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*For a general treatment of this see Art. 22.

93. Line of Force and Equipotential Surface—The space, surrounding a massive body or a system of massive bodies in which measurable gravitational attractions take place, is called the *gravitational field* due to the body, or the system of bodies. A *line force* is a curve, such that the tangent at any point of it represents the direction of the resultant force at that point. A surface, at any point on which the gravitational potential has one and the same value, is called an *equipotential* (gravitational) *surface*. Thus, no work is done in carrying a unit mass from one point to another on an equipotential surface.

At any point in a gravitational field, a line of force and an equipotential surface cut each other at right angles, as otherwise there will be a component of the force along the equipotential surface at that point and work will be done in taking a unit mass along the equipotential surface in the neighbourhood of the point, which is contrary to the definition of an equipotential surface.

It is evident from the definition of a line of force that two lines of force cannot cut each other; for, in that case there will be *two resultant* directions of force *along* the *tangents to the two lines of force drawn* at the point where they intersect. This is contrary to the definition of a line of force. Also two equipotential surfaces cannot cut each other, as otherwise there will be *two resultant* directions of force at the point of intersection along the normals to the equipotential surfaces drawn at the point where they cross. This is not consistent with the definition of a line of force.

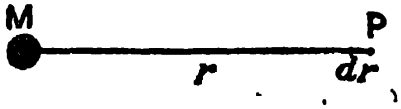

Fig. 52

94. Gravitational Potential at a point due to a Point Mass—Consider a particle of mass M (fig. 52). The attraction (on a unit mass) at a point P situated at a distance r from the particle$=G\frac{M}{r^2}$. The work done in carrying the unit mass towards the particle through a small distance dr, over which the attractive force may be supposed to be constant

$$= G\frac{M}{r^2}.dr.$$

Hence, the work performed in carrying the unit mass from an infinite distance up to the point P, which measures the potential at P due to the massive particle

$$= \int_{\infty}^{r} G\frac{M}{r^2}\,dr = G\left[-\frac{1}{r}\right]_{\infty}^{r} = -G\frac{M}{r}$$

The negative sign shows that no work is required to be done by the external agency but the work will be done by the gravitational attraction.

95. Potential at a point due to a thin uniform Spherical Shell.

(1) *The point P is external to the shell.*

In fig. 53, P is a point external to the shell of radius a. O represents the centre of the shell. Let the distance of P from O be r and the surface density (mass per unit area) of the shell be m.

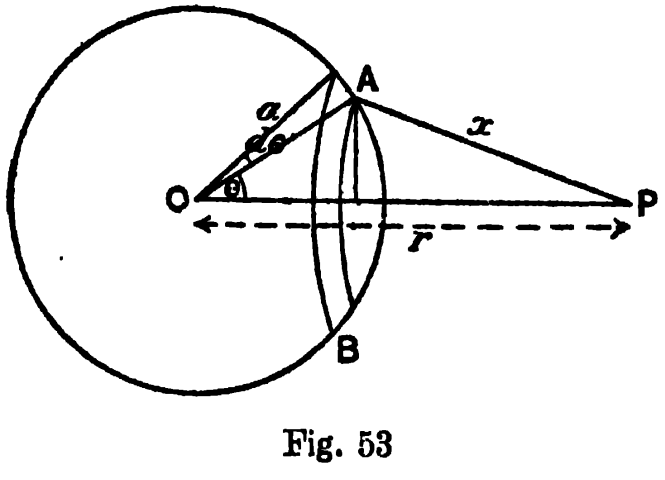

Fig. 53

Consider a circular ring AB of the shell of small width having its centre on OP.

Let the radius of the ring subtend an angle θ and the width of the ring subtend an angle $d\theta$ at O.

The radius of the ring $= a \sin\theta$ and its width $= a d\theta$.

The surface-area of the ring = its circumference × width

$$= 2\pi a \sin\theta . a d\theta = 2\pi a^2 \sin\theta d\theta.$$

The mass of the ring $= 2\pi a^2 \sin\theta d\theta . m$

If x represents the distance of P from any point of the ring the potential at P due to the ring (from Art. 94)

$$= -G\frac{2\pi a^2 \sin\theta\, d\theta . m}{x} \quad \ldots \quad \ldots \quad (1)$$

This expression, when integrated over the whole shell, will give the potential at P due to the shell. It contains two variables, θ and x. To eliminate one, we may utilise the relation

$$x^2 = a^2 + r^2 - 2ar\cos\theta.$$

Differentiating,

$2x\, dx = 2ar \sin\theta\, d\theta$ (since a and r are constants).

$$\therefore \quad \sin\theta\, d\theta = \frac{x dx}{ar}$$

Substituting for $\sin\theta\, d\theta$ in (1), the potential at P due to the ring

$$= -G\frac{2\pi a^2 x\, dx . m}{ar . x} = -G\frac{2\pi m a}{r}dx \quad \ldots \quad (2)$$

$\therefore$ the potential at P due to the shell

$$= \int_{x=r-a}^{x=r+a} -G\frac{2\pi m a}{r}\, dx = -G\frac{2\pi m a}{r}\Big[\, x \,\Big]_{r-a}^{r+a}$$

$$= -G\frac{2\pi m a}{r} . 2a = -G\frac{4\pi a^2 m}{r}$$

Again, $4\pi a^2$ is the surface-area of the shell Hence, $4\pi a^2 m$ is the mass of the shell. Representing the mass of the shell by M, the potential at P due to the shell

$$= -G\frac{M}{r}$$

But the potential at P for a particle of mass M situated at O, is of the same magnitude (Art. 94)

Hence, for an external point a spherical shell behaves, as if its mass were concentrated at its centre.

If *the point P is on the shell*, $r=a$. Hence, the potential at P

$$=-G\frac{M}{a}$$

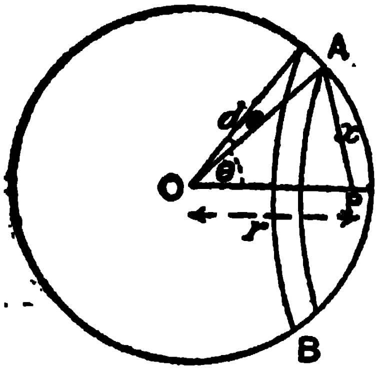

Fig. 54

(2) *The point P is inside the shell.*

As before the potential at P (fig. 54) due to the ring AB

$$=-G\frac{2\pi ma}{r}dx \ldots \left(\begin{array}{c}\text{equation 2}\\ \text{above}\end{array}\right)$$

The petential at P due to the shell

$$=\int_{x=a-r}^{x=a+r} -G\frac{2\pi ma}{r}dx=-G\frac{2\pi ma}{r}\Big[\ x\ \Big]_{a-r}^{a+r}$$

$$=-G\frac{2\pi ma}{r}\,.\,2r=-G.4\pi ma$$

$$=-G\frac{4\pi a^2 m}{a}=-G\frac{M}{a}$$

where M is the mass of the shell.

This expression is independent of r (the distance of P from the centre of the shell).

Thus, the potential at a point inside the shell is a constant which is equal to the potential of the shell.

96. Attraction at a point due to a thin uniform Spherical Shell.

(1) *The point P is external to the shell.*

The attraction (force on unit mass) at P can be indirectly calculated from the expression for potential at P.

From symmetry, the attraction at P will be along the line PO (fig. 53) and since the attraction is the *space rate* of potential (Art. 92), the attraction at P

$$=\frac{d}{dr}\left(-G\frac{M}{r}\right)=G\frac{M}{r^2}$$

Thus, the attraction at an external point, due to a thin uniform *spherical shell is the same, as if the whole mass of the shell were concentrated at its centre.*

If the point P is on the shell, $r=a$, and the attraction at P

$$=G\frac{M}{a^2}$$

(2) *The point P is inside the shell.*

When P is inside the shell (fig. 54), from symmetry the attraction at P, if any, will be along the line PO. Its magnitude can be determined by the *space rate* of potential (Art. 92). Therefore, the attraction at P

$=\frac{d}{dr}\left(-G\frac{M}{a}\right)=0$ (since the expression within brackets is independent of r).

Thus, the attraction at a point inside a thin uniform spherical shell is zero.

97. Independent Calculation for Attraction—The expression for attraction at external and internal points due to a thin uniform spherical shell can be calculated without assuming the expressions for potential as follows :—

(1) *The point P is external to the shell.*

Consider a ring AB (fig. 53) as in Art. 95 (1) and an *element* of the ring at A of area s. The mass of this element$=s.m$. The attraction at P due to this element is along PA and is of magnitude

$$G\frac{sm}{x^2}.$$

Resolving this attraction at P into two components, along and perpendicular to PO, we see that the component perpendicular to PO is neutralised by an equal and opposite component of the attraction at P due to a diametrically opposite element at B of the same area. Hence, the resultant of the attractions due to both these elements is along PO. The entire ring can be divided into such pairs of elements, each pair of which mutually neutralises the components perpendicular to PO. For any element, the resultant attraction is thus along PO and is of magnitude

$G\frac{sm}{x^2}\cos\phi$, where ϕ is the angle APO.

The attraction at P due to the ring

$$=G\frac{2\pi a^2\sin\theta d\theta\, m}{x^2}\cos\phi\text{.........(1)}\ \left(\begin{matrix}\text{replacing } s \text{ by the}\\ \text{area of the ring}\end{matrix}\right)$$

To express ϕ in terms of the variable x, we see

$$a^2 = x^2 + r^2 - 2xr\cos\phi$$

$$\therefore \quad \cos\phi = \frac{x^2 + r^2 - a^2}{2xr}$$

$\therefore$ the attraction at P due to the ring, from equation (1),

$$= G\frac{2\pi a^2 \sin\theta d\theta . m}{x^2} . \frac{x^2 + r^2 - a^2}{2xr}$$

Replacing $\sin\theta d\theta$ by $\frac{xdx}{ar}$, the attraction at P due to the ring

$$= G\frac{\pi am}{r^2} . \frac{x^2 + r^2 - a^2}{x^2} dx \quad \dots \quad \dots \quad (2)$$

The attraction at P due to the shell

$$= \int_{x=r-a}^{x=r+a} G\frac{\pi am}{r^2} . \frac{x^2 + r^2 - a^2}{x^2} dx$$

$$= G\frac{\pi am}{r^2}\left\{\int_{r-a}^{r+a} dx + (r^2 - a^2)\int_{r-a}^{r+a}\frac{dx}{x^2}\right\}$$

$$= G\frac{\pi am}{r^2}\left\{2a + 2a\right\} = G\frac{4\pi a^2 m}{r^2} = G\frac{M}{r^2}.$$

(2) *The point P is inside the shell.*

The attraction at P due to ring AB (fig. 54) as in equation (2) above

$$= G\frac{\pi am}{r^2} . \frac{x^2 + r^2 - a^2}{x^2} dx$$

The attraction at P due to the shell

$$= \int_{x=a-r}^{x=a+r} G\frac{\pi am}{r^2} , \frac{x^2 + r^2 - a^2}{x^2} dx$$

$$= G\frac{\pi am}{r^2}\left\{\int_{a-r}^{a+r} dx + (r^2 - a^2)\int_{a-r}^{a+r}\frac{dx}{x^2}\right\}$$

$$= G\frac{\pi am}{r^2}\left[\ 2r - 2r\ \right] = 0.$$

98. Potential at a point due to a Homogeneous Solid Sphere—

(1) *The point P is external to the sphere.*

Let the sphere be of uniform volume density (mass per unit volume); it may be supposed to be divided into thin uniform concentric spherical shells. The potential at the external point P due to any of these shells may be calculated by supposing the mass of the shell to be concentrated at its centre (Art. 95, 1). Thus, summing up for all the shells the potential at the external point P due to the sphere may be calculated on the supposition that the mass of the sphere is concentrated at its centre. The potential at P at a distance r from the centre of the sphere

$$= -G\frac{M}{r} \text{ where } M = \text{the mass of the sphere.}$$

(2) *The point P is inside the material of the sphere.*

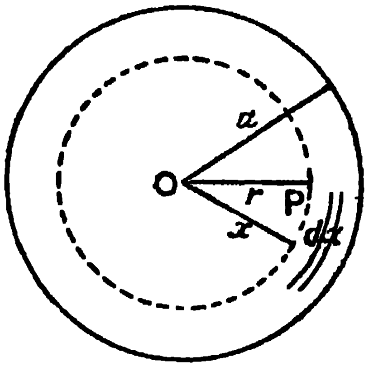

Fig. 55

Let the point P be within the sphere (fig. 55) at a distance r from its centre. Imagine a concentric spherical surface through P. The potential at P arises out of the inner sphere and the outer thick spherical shell. To find the potential at P due to the inner sphere, we see that P is situated outside it and therefore the potential can be calculated by supposing the mass of the inner sphere to be concentrated at its centre. If ρ be the volume density of the sphere, the mass of the inner sphere $= \frac{4}{3}\pi r^3\rho$ and the potential at P due to

$$\text{the inner sphere} = -G\frac{\frac{4}{3}\pi r^3\rho}{r} = -\frac{4}{3}\pi r^2\rho G \quad \ldots \quad (1)$$

To calculate the potential at P due to the outer thick shell, imagine a thin concentric shell in it of radius x and thickness dx. The volume of the thin shell = the area of its surface × thickness $= 4\pi x^2 . dx$ and its mass $= 4\pi x^2 dx\rho$

The potential at P due to this thin shell

$$= -G\frac{4\pi x^2 dx\rho}{x} (\text{Art. } 95, 2) = -G.4\pi x dx\rho$$

∴ the potential at P due to the outer thick shell

$$= \int_{x=r}^{x=a} -G \cdot 4\pi x dx \rho = -2\pi G\rho\,(a^2 - r^2) \qquad \ldots \qquad (2)$$

Hence, the potential at P due to the sphere, which is the sum of (1) and (2)

$$= -\frac{4}{3}\pi r^2 \rho G - 2\pi G\rho\,(a^2 - r^2) = -\frac{2}{3}\pi\rho G(3a^2 - r^2)$$

$$= -GM\frac{3a^2 - r^2}{2a^3}. \quad \left(\text{as } \rho = \frac{M}{\frac{4}{3}\pi a^3}\right)$$

99. Attraction at a point due to a Homogeneous Solid Sphere.

(1) *The point P is external to the sphere.*

As in Art. 98, for an external point the mass M of the sphere may be supposed to be concentrated at its centre.

Thus, the attraction at P, at a distance r from the centre of the sphere $= G\dfrac{M}{r^2}$.

(2) *The point P is internal to the sphere.*

Imagine a concentric spherical surface through P as in Art. 98, 2 (fig. 55). The attraction at P due to the inner sphere can be calculated by supposing the mass of the inner sphere to be concentrated at its centre. Therefore, if the distance of P from the centre is r, the mass of the inner sphere $= \frac{4}{3}\pi r^3\rho$ and the attraction at P due to the inner sphere

$$= G\frac{4\pi r^3\rho}{3r^2} = \frac{4}{3}\pi G r\rho.$$

The attraction at P due to the outer thick shell is zero; because, the thick shell may be divided into thin concentric shells, the attraction due to each of which at P is zero as P is internal to it (Art. 96, 2).

∴ The attraction at P due to the sphere $= \frac{4}{3}\pi G r\rho = GM\dfrac{r}{a^3}$.

103. Potential at a point due to rod.

Let AB be a rod of length L (fig. 56) and linear density m. Draw OP perpendicular to the rod from its middle point O. The point P is at a distance r from O.

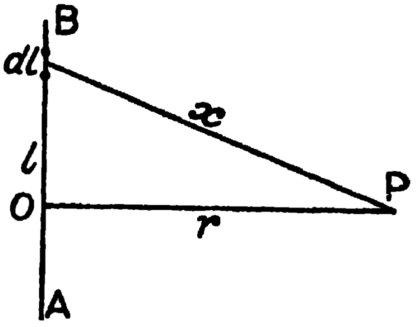

Fig. 56

Consider an element of length dl of the rod at a distance l from O. Let the distance of P from the element be x. The potential at P due to the element $= -\dfrac{G\,mdl}{x}$ ... ... (Art. 94)

$$= -\frac{G\,m\,dl}{\sqrt{l^2+r^2}}.$$

The potential at P due to the rod

$$= -2\int_0^{L/2}\frac{G\,m\,dl}{\sqrt{l^2+r^2}} = -2Gm\Big[\log_e\big(l+\sqrt{l^2+r^2}\big)\Big]_0^{L/2}$$

$$= -2Gm\left\{\log_e\left(\frac{L}{2}+\sqrt{\frac{L^2}{4}+r^2}\right)-\log_e r\right\}$$

104. Attraction at a point due to a Rod.

AB (fig. 56) is a rod of linear density m. The attraction at P at a distance x due to an element dl of the rod at a distance x from it is $\dfrac{G\,mdl}{x^2}$ towards the element. The component of this along PO drawn perpendicular to the rod is $\dfrac{G\,mdl}{x^2}\cos\theta$ where θ is the angle which the line joining P to the element makes with PO. The other component $\dfrac{G\,mdl}{x^2}\sin\theta$ perpendicular to OP is neutralised by contribution from an equal element on the other side of O at an equal distance from it. This is true for any equal pair of elements similarly situated with respect to OP.

Hence, the attraction at P due to an element of length dl of the rod is along PO and is of magnitude

$$\frac{G\,m\,dl}{x^2}\cos\theta=\frac{G\,m\,dl}{l^2+r^2}\cos\theta$$

Also $l=r\tan\theta \quad \therefore \quad dl=r\sec^2\theta\,d\theta$

$\therefore$ attraction at P due to the element dl

$$=\frac{G\,m\,r\sec^2\theta\cos\theta\,d\theta}{r^2(1+\tan^2\theta)}=\frac{Gm}{r}\cos\theta\,d\theta$$

Attraction at P due to a rod of infinite length

$$=2\int_0^{\pi/2}\frac{Gm}{r}\cos\theta\,d\theta=\frac{2Gm}{r}.$$

105. Potential and Attraction at a point on the axis of a thin Uniform Circular Disc.

Consider a circular disc of mass M and radius a with D as its centre (fig. 57). The point P is on the axis of the disc at a distance r from D. Imagine an annulus on the disc bounded by radii x and $x+dx$. Area of the annulus$=2\pi x dx$. Area of the disc $=\pi a^2$

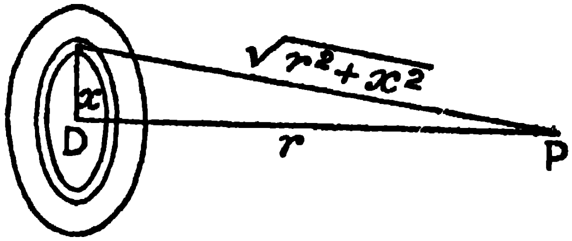

Fig. 57

$\therefore$ mass of the annulus$=\frac{M}{\pi a^2}\cdot 2\pi x\,dx=\frac{2M}{a^2}x\,dx.$

Potential at P due to the annulus$=-G\frac{2M}{a^2}x\,dx.\ \frac{1}{\sqrt{r^2+x^2}}$

as $\sqrt{r^2+x^2}$ is the distance of P from any point of the annulus. (Art. 94).

Potential V at P due to the disc $= -G\frac{M}{a^2}.\int_0^a \frac{2x\,dx}{\sqrt{r^2+x^2}}$

$$=\frac{2GM}{a^2}(r-\sqrt{r^2+a^2})$$

Intensity of attraction at $P=\frac{dV}{dr}=\frac{2GM}{a^2}\left(1-\frac{r}{\sqrt{r^2+a^2}}\right)$

106. Variation in the Weight of a Body at Different Distances from the Centre of the Earth—

If the earth is supposed to be approximately spherical in shape and of uniform density, the weight of a body within the earth at various depths will increase from a zero value at the centre to a value $G\frac{Mm}{R^2}$ on the surface of the earth ; where M and m are the respective masses of the earth and the body and R the radius of the earth. The variation in weight with the distance of the body from the centre of the earth will satisfy a linear relation (Art. 99, 2) as shown in fig. 58.

As the body is taken further and further above the surface of the earth, its weight will decrease satisfying a relation

$$Wr^2=\text{constant}$$

where W=the weight of the body at a distance r from the centre of the earth (Art. 99, 1). The variation is shown in fig. 58. Thus, the weight of a body is a maximum on the surface of the earth and it falls bothways vanishing at the centre of the earth and at infinity. Because of the spheroidal shape of the earth and want of homogeneity in its constituents, the actual variation in weight

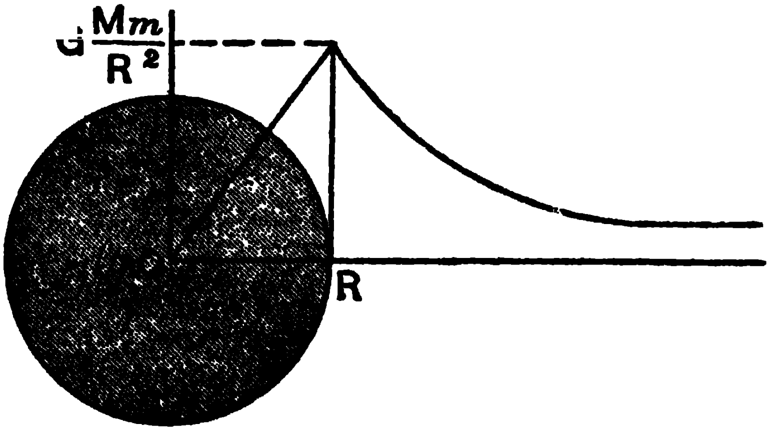

Fig. 58

of a body at different distances deviates from the above calculation.

107. Gravitational Constant : Mass and Density of the Earth—The attractive force exerted by the earth on a body near its surface is given by $G\frac{Mm}{R^2}$, where M and m represent the masses of the earth and the body, R the radius of the earth and G the gravitational constant. This attractive force is the weight mg of the body.

$$\therefore \quad mg = G\frac{Mm}{R^2}, \text{ or, } g = G\frac{M}{R^2} \quad \dots \quad \dots \quad (1)$$

Since g, the acceleration due to gravity, can be accurately determined at a place by the compound pendulum, and R the radius of the earth is known, a determination of G suffices to find out M. Also if D is the mean density of the earth,

$$M = \tfrac{4}{3}\pi R^3 D \quad \dots \quad \dots \quad \dots \quad (2)$$

$$\therefore \text{ From (1) } g = \tfrac{4}{3}\pi\, GRD \text{ or, } D = \frac{3g}{4\pi GR} \quad \dots \quad \dots \quad (3)$$

Hence, from a determination of G, the mass of the earth M can be found from relation (1) and the mean density D of the earth from (3). An experiment to determine G is sometimes called "*Weighing the Earth*", although the term is inaccurate in the sense that the earth cannot properly be said to have any weight as the term weight is applied to the attraction which the earth exerts upon a body.

The experiments to determine G and D fall primarily into two classes.—(1) In one, *the mountain method*, some natural mass, such as a mountain, is selected, the mass and centre of gravity of which are known from survey and mineralogical examination of the mountain. Its attraction on a plumb line is compared with the attraction of the earth. (2) In the other, *the laboratory class of experiments*, the attraction between two masses is determined and hence G is found out.

108. Measurement of the Mass of the Earth—*The mountain method*—Suppose a plumb bob is suspended near the side of a mountain. It will be attracted by a horizontal force due to the attraction of the mountain and by a vertical force due to the remainder of the earth. The supporting cord of the bob will take up the direction of the resultant of these two forces and in consequence will be deflected from vertical position through an angle θ, where

$$\tan\theta = \frac{\text{Horizontal force exerted by the mountain}}{\text{Vertical force exerted by the earth}}$$

If m be the mass of the mountain and d the distance of its centre of gravity from the plumb bob on the same horizontal line, the horizontal force on the bob exerted by the mountain $=G\dfrac{mm'}{d^2}$. where m' is the mass of the plumb bob. Similarly, if M be the mass of the remainder of the earth and R the distance of the centre of the earth from the plumb bob, the vertical force exerted by the earth $=G\dfrac{Mm'}{R^2}$

$$\therefore \quad \tan\theta = \frac{G\dfrac{mm'}{d^2}}{G\dfrac{Mm'}{R^2}}, = \frac{R^2m}{d^2M}$$

The first experiment to measure the deflection of a plumb line was conducted by Bouguer in 1740, the mountain was Chimborazo in the Andes. The next experiment was made by Maskelyne in 1774, the mountain was Schiehallion in Perthsire. The deflection of the plumb line in both the experiments was measured from the apparent shift of a star observed by a telescope* and the mass of earth was calculated from known values of m, d and R. From this the mean density of the earth and the value of the gravitational constant were found out.

109. Airy's Mine Experiment—Airy in 1854 determined the value of g at the top and bottom of the Harton coal-pit, near Sunderland by means of a pendulum.

The depth of the mine shaft was measured. Samples of the rocks through which the shaft was bored were assayed and the density of the surface constituents was calculated. The mean density of the earth was thus found out.

110. The Cavendish Experiment—The laboratory class of experiments to measure the force of attraction between two massive balls was suggested by Rev. John Michell and after his death the experiment was carried out by Cavendish in 1798.

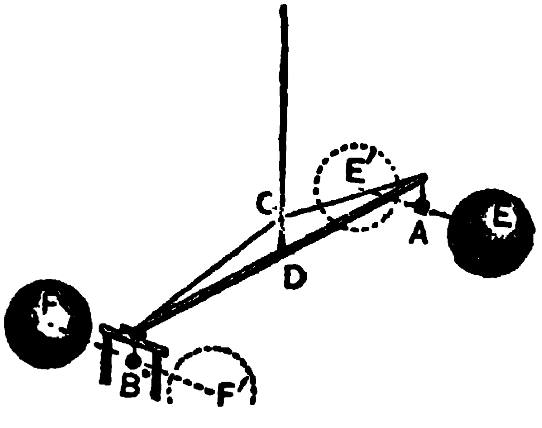

Fig. 59

Two balls A and B (fig. 59) of lead, each 1 inch in radius, were suspended by short wires from the ends of a light rod 6 feet long. To strengthen the rod without unduly increasing its mass, its ends were tied by a wire to a short stiff upright DC through its middle point D. The rod was suspended horizontally by a vertical wire fixed at one of its ends to the upright at C, the other end of the wire being clamped in a torsion head. Two large lead balls E and F, each of radius 4 inches, were placed on opposite sides of the small balls A and B at equal distances from them respectively. The centres of all the balls were in the same horizontal plane. Attraction followed between the balls E and A. Also a similar attraction between the balls F and B. These attractive forces constituted a gravitational couple and the rod in consequence was turned through. The angle of turning was measured by a scale placed on a stand near one end of the rod and a vernier carried by the end of the rod without touching the scale.

In an actual observation the reading of the vernier in the deflected position when the attracting balls were at E and F was [illegible] The attracting balls were then moved to E' and F' such [illegible] the distances between the centres of the attracting and [illegible] balls were the same as before.

The attractive forces on the balls A and B now acted in the opposite direction and the balls A and B were set to oscillations. The resting position of the vernier was calculated from its three successive turning positions along the scale as in the oscillation-method of weighing. The angle through which the rod turned between the two experiments was evidently double the angle θ through which it would turn from its equilibrium position in either experiment.

If M and m represent the masses of one of the attracting and attracted balls respectively and d the distance between their centres, the force of attraction on A or B

$$=G\frac{Mm}{d^2}$$

If the length of the rod be l, the moment of the couple formed by the two forces on A and B

$$=Gl\frac{Mm}{d^2}$$

This turning couple on the balls A and B is balanced by the controlling couple due to torsion in the suspending fibre. If τ

represents the moment of the torsional couple per unit twist, the moment of the torsional couple for a twist $\theta = \tau\theta$, and for equilibrium

$$Gl\frac{Mm}{d^2}=\tau\theta$$

M, m, d, l and θ were measured in the experiment. The attracting balls were then removed and τ was calculated from the period of torsional oscillation of the suspending system. If T is the period and I, the moment of inertia of the suspending system,

$$T=2\pi\sqrt{\frac{I}{\tau}} \quad \ldots \quad \text{(Art. 75,1)}$$

Calculating I and T, τ was found out and hence the value of G was determined. Cavendish found from the mean of 29 observations—

$$G=6{\cdot}754\times10^{-8} \text{ C. G. S. units}$$

$$\text{and } D=5{\cdot}448 \text{ gms. per c. c.}$$

Cavendish kept his apparatus in a closed room which [illegible] not enter during his experiment. The apparatus was kept [illegible] a gilt-glass case which served to keep the temperature constant inside to prevent air currents and also to avoid electro-static attractions from outside. All the observations and measurements were made from outside the glass case.

111. Boys' modification of the Cavendish Experiment.

In the Cavendish experiment the accuracy was limited owing to the following factors :—

(1) The suspension was rather thick. This made τ comparatively large and the deflection consequently small.

(2) The measurement of the deflection by vernier was not capable of great accuracy.

(3) Temperature-gradients and the consequent movement of air could not be avoided in the large apparatus.

(4) The appreciable attractive force exerted by E on B along BE if resolved along and normal to the length of the rod, the latter component will act opposite to the direction in which F attracts B. This component together with an equal and opposite component of the force exerted by F on A produce a counter-gravitational couple which still further reduced θ.

In the apparatus used by Professor C. V. Boys (1895), shown

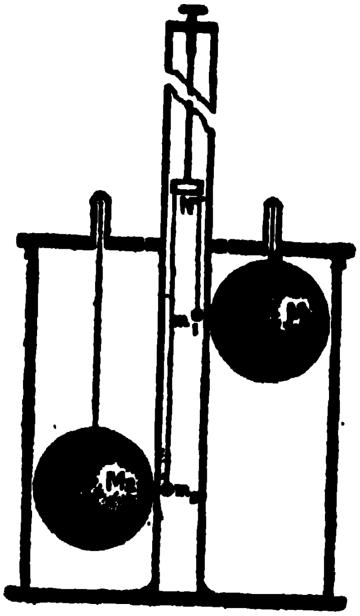

Fig. 60

in Fig. 60, the attracted balls m_1 and m_2 made of gold, were suspended by two quartz fibres from the ends of a small mirror N. The mirror was suspended by a quartz fibre from a torsion head. The quartz suspension reduced τ and at the same time it is sufficiently strong This suspension system is hung inside a glass tube, of internal diameter about 3·8 cm. and is thus protected from draughts. Further, the air pressure inside the tube can be reduced to any desired value. The attracting balls M_1 and M_2 were of lead about 10·8 cm. in diameter. The deflection of the mirror was measured by reflecting a beam of light from it. The apparatus was fairly of small size. In order to minimise the counter-gravitational couple, the balls M_1 and m_1 were suspended with their centres on a horizontal plane, six inches above the horizontal plane in which the centres of M_2 and m_2 lay. Each lead sphere exerts an attraction on each gold sphere but due to difference in levels between the two pairs of attracting and attracted spheres, the attraction of a lead sphere on that gold sphere whose centre is on a level with its own is far greater than that on the other. The effect of the gravitational attractions is to produce a deflecting torque. The controlling torque is due to the twist produced in the central suspending fibre. The suspension system comes to rest when the two torques balance. The position of the light spot on the scale is noted. The lead spheres are then moved to new and similar positions on the opposite sides of the gold spheres so as to produce the same deflecting torque in the opposite direction and the change in the scale reading is noted. The method of operation was analogous to that of the Cavendish method.

If d be the shift of the light spot on a scale placed at a distance D from the mirror, the angular shift of the reflected beam is $\frac{d}{D}$ and that of the mirror is $\frac{d}{2D}$ as the reflected beam turns through twice the angle through which the mirror turns. Further, this angular shift of the spot from one deflected position to another is due to the reversal of the gravitational couple. Hence, the angular shift θ of the mirror or the suspension system from undeflected position due to the couple exerted by the lead balls is half of it or $\theta = \frac{d}{4D}$……(1)

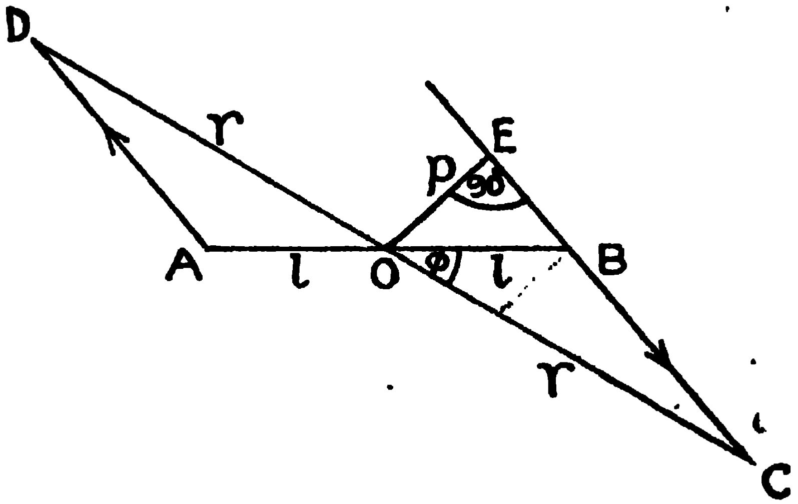

Fig. 61

Let M and m be the masses of either of the lead and gold spheres. In fig. 61 A and B are the centres of the gold spheres and C and D are those of the lead spheres. When equilibrium has been reached. Neglecting attractions between spheres at different levels, the attraction of the lead sphere at C is along BC and is of magnitude $G\frac{Mm}{BC^2}$. The moment of this about the middle point O of the mirror at which the suspension fibre is attached is $G\frac{Mm}{BC^2}p$ where p is the perpendicular distance from O on CB produced. An equal gravitational attraction exerted by the lead sphere at D on the gold sphere at A along AD produces an equal moment of the same sense on the suspension system, so that the total turning torque on the system

$$=2G\frac{Mm}{BC^2}\cdot p$$

Again $p.\ BC = 2\times$area of the triangle $OCB = lr\sin\phi$ where $OB = l$, $OC = r$ and $\angle BOC = \phi$

Hence the total deflecting torque is

$$=2G\frac{Mm}{BC^3}\cdot p.\ BC = 2G\frac{Mm}{BC^3}lr\sin\phi$$

1420

Also from ΔOBC, $BC^2 = l^2 + r^2 - 2lr\cos\phi$.

Hence the total deflecting torque

$$=2G\frac{Mmlr\sin\phi}{(l^2+r^2-2lr\cos\phi)^{3/2}} \qquad \ldots(2)$$

If τ be the torque per unit twist of the central suspending fibre and θ the twist as calculated from (1) the controlling torque is $\tau\theta$. Equating the deflecting torque (2) and controlling torque $\tau\theta$ for equilibrium,

$$2G\frac{Mmlr\sin\phi}{(l^2+r^2-2lr\cos\phi)^{3/2}}=\tau\theta \qquad \ldots(3)$$

Removing the lead spheres and setting the suspension system to torsional oscillations, τ is calculated from the period of oscillation $T=2\pi\sqrt{\frac{I}{\tau}}$ where I is the moment of inertia of the suspension system. The value of G is then calculated from equation (3).

The advantage of the method lies in the use of thin quartz fibres which are extremely sensitive and which retain their elastic properties. The great reduction in the size of apparatus almost eliminates the possibility of temperature changes in different parts of a bulky apparatus and the consequent convection currents of the inside air.

The method of operation was similar to that of the Cavendish method. The results obtained were

$G=6{\cdot}658\times10^{-8}$ C. G. S. units
and $D=5{\cdot}527$ gms. per c.c.

112. Poynting's Balance Experiment—

At the suggestion of Von Jolly, the ordinary balance was employed to find G and D by Poynting in 1891. Two lead spheres, each of mass 50 lbs. were suspended from the ends of the beam of a sensitive balance. An attracting sphere of lead weighing 350 lbs. was at first brought under one of the suspending spheres and placed at a certain distance below it. The tilt of the beam was noted by the deflection of the pointer. The observation was repeated by placing the large sphere below the mass suspended from the other end of the beam and the mean deflection of the pointer of the balance was noted.

Let M be the mass of the large sphere placed with its centre at a distance d below the centre of any of the attracted spher of mass m and let mean deflection of the pointer of the tw observations be θ. If $2l$ be the length of the balance beam, moment of the deflecting force about the fulcrum of the balance is given by

$$G\frac{Mm}{d^2}l = K\theta \qquad \dots \qquad \dots \qquad (1)$$

where K is a constant.

A centigram rider was then moved through a distance x along the beam to obtain a deflection ϕ of the same order of θ

$$\therefore \quad \cdot 01\, g\, x = K\phi \qquad \dots \qquad \dots \qquad (2)$$

From (1) and (2), $\therefore \dfrac{G\frac{Mm}{d^2}l}{\cdot 01\, g x} = \dfrac{\theta}{\phi}$.

This relation gives the value of G and hence of D, the mean density of the earth.

Poynting placed his balance in an underground room and enclosed it to diminish disturbances due to air currents. All manipulations were made by mechanisms controlled by rods from the outside.

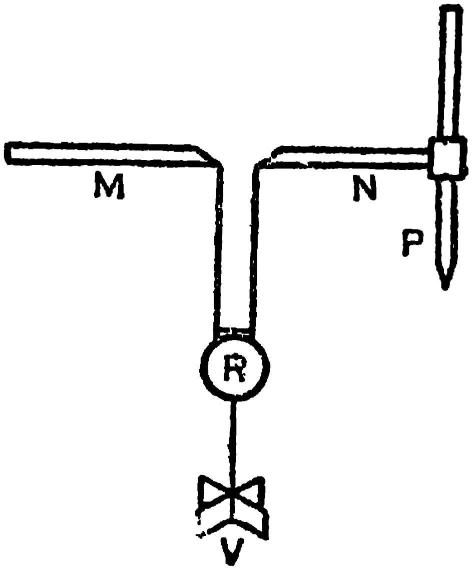

Fig. 62

The ratio $\frac{\theta}{\phi}$ was measured by a device suggested by Lord Kelvin. The balance pointer P (fig. 62) was attached to an arm N at right angles to it. There was a second arm M fixed in position at a very short distance from N. A light mirror R was suspended by two strings attached to the ends of the arms. To avoid the disturbing effect of air, two crossed vanes V were suspended from the mirror and kept immersed in oil. Any angular deflection of the pointer turned the mirror through an angle 150 times as great. This was noted by observing the displacement on a scale of a spot of light reflected from the mirror. The ratio $\frac{\theta}{\phi}$ is evidently the same as that between the displacements of the light spot in the two cases.

The accepted values are

$G = 6{\cdot}66 \times 10^{-8}$ C.G.S. units

$D = 5{\cdot}52$ gms. per c.c.

113. Qualities of Gravitation—

(1) *Permeability*—In magnetism and electricity, the laws of interaction between magnetic poles or electric charges have a close resemblance with the Newtonian laws of gravitation. In magnetism and electricity, however, the attraction between magnetic poles or electric charges depends upon the nature of the intervening medium. An experiment was conducted by Austin and Thwing in 1897, to test whether the gravitational forces of attraction between two bodies were affected by the change of the intervening medium. The experiment consisted of a modified form of Boys' experiment in which slabs of various materials were interposed between the attracting and attracted balls. No change in the value of *G* was, however, detected.

(2) *Selectivity*—In magnetic and electric attractions, some kind of matter is found tc be more attracted than others.

According to the Newtonian laws of gravitation, the nature of attraction is determined by the magnitude of the attracting masses only and does not depend on the nature of them. From magnetic and electric analogy, it might be possible that some kind of matter is attracted more in proportion to its mass than others. A supposition of this description was made to explain the formation of comets' tails, some matter being supposed to be repelled by the sun. Bessel's experiments with hollow pendulum (1830), though not conclusive, prove that the earth is not selective. The explanation of comets' tails is now given on an electrical basis.

(3) *Directivity*—The physical properties of a crystal are different in different directions. They possess different coefficients of thermal expansion, different heat conductivities and different velocities of light-transmission in different directions. It might be similarly expected that the lines of gravitational force spread out unequally in different directions from a crystalline sphere. To test this, Dr. Mackenzie in 1895 conducted an experiment in which an apparatus similar to Boys' torsion balance was used. The attracted spheres were small calc-spar crystal spheres. He measured the forces of attraction between the crystalline sphere and lead sphere when the axes of the crystal were turned in different directions.

There was, however, no evidence of variation in the attractive force with the variation of axial direction.

114. Flux of Gravitational Intensity—

Let P be a point in a gravitational field where the gravitational intensity F is along PA (fig. 63). Through P draw a closed surface S which has a unique outward drawn normal at every point. Let dS be an infinitesimally small portion of the surface surrounding P and PN an outward drawn normal to it. If ϕ be the angle between PA and PN, the component of F along the outward drawn normal$=F\cos\phi=F_n$ (say).

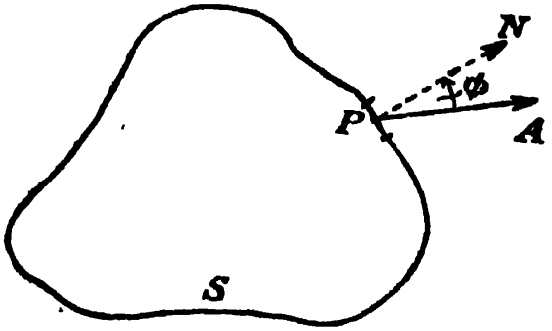

Fig. 63

The quantity $F_n dS$ or $F\cos\phi\, dS$ is called the flux of gravitational intensity across the surface dS. Similarly, $\int F_n dS$ or, $\int F\cos\phi\, dS$ where the integration extends over the whole surface S, is called the flux of gravitational intensity over the surface S.

115. Gauss' Theorem—It states that the flux of gravitational intensity across a closed surface which has a unique outward drawn normal at every point is $-4\pi G$ times the total mass enclosed by that surface.

or, $\int F_n dS=-4\pi G\Sigma(m)$ where $\Sigma(m)$ is the total mass inside the closed surface.

To, prove the theorem, let there be a particle of mass m at O (fig. 64) and S any closed surface surrounding O. Let BC be an infinitesimally small portion of the surface of area dS round a point D on the surface. If r be the distance of D from O, the gravitational intensity at D due to m is $G\frac{m}{r^2}$ along DO *or* $-G\frac{m}{r^2}$ along ODA. If ϕ be the angle between DA and DN, the normal

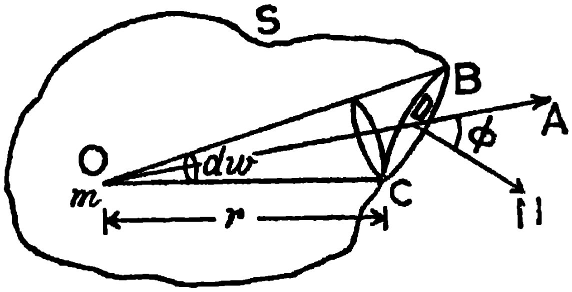

Fig. 64

drawn to the surface at D, the normal component of the intensity at $D = F_n = -G\frac{m}{r^2}\cos\phi$

The flux of gravitational intensity across the surface BC $= F_n dS = -G\frac{m}{r^2}\cos\phi\, dS = -Gm\,d\omega$

where $d\omega$ is the solid angle subtended at O by the area dS.

Hence, the flux of gravitational intensity across the closed area $S = \int -Gm\,d\omega = -4\pi Gm$.

If there be several particles within the closed surface, the theorem is applicable to each of them and the flux of gravitational field for all the masses $= -4\pi G\Sigma(m)$ where $\Sigma(m)$ is the mass of the material system enclosed by the surface.

116. Laws of Gravitation and Laws of Planetary Motion—

Long before Newton's discovery of the laws of gravitation Kepler discovered the laws of planetary motion which are :—

(1) The planets describe ellipses with the sun at one focus. ✓

(2) The area traced out by a radius vector drawn from the sun to the planet is directly proportional to the time.

(3) The squares of the periodic times are proportional to the cubes of the mean distances of the planets from the sun.

Newton showed in his *principia* that if a body moves along a curved line in a plane and if it describes areas about a point either fixed or moving in a straight line by radii drawn to that point such that the areas are proportional to the times of describing them, then the body is acted upon by a centripetal force towards that point. This explains the first two laws of Kepler and shows that there is a force of attraction exerted by the sun upon the planets.

Newton further showed that if a body describes an ellipse under the action of a force directed towards the focus of the ellipse, the force must vary inversely as the square of its distance from the focus. He also showed that if a body revolves about another such that the centripetal force varies inversely as the square of the distance, then the square of the periodic time is proportional to the cube of the major axis of the ellipse.

The astronomical calculations based upon the law of gravitation agree so well with observed facts that its truth is well established. As an illustration, the planet Uranus discovered by by Herschel in 1781 showed certain deviations from the law of gravitation with regard to its motion. In 1845 Adams showed that the refractory nature of Uranus could be explained if there were an outer planet yet unknown in a specified position. Following this prediction Galle discovered the new planet Neptune in practically the same position. Again, the times of revolution of certain comets calculated from the law of gravitation have been found to be true. Thus, for astronomical calculations Newton's law of gravitation has been found adequate. There are, however, some minute discrepancies in several cases between calculation and observation which have been explained by the Theory of relativity of Einstein.

The law is *universal as far as huge interplanetary* distances or small terestrial distances are concerned. The law appears to break down for inter-molecular distances which are of the order of 10^{-7}.cm.

Examples

1. A straight tunnel is bored through the earth which may be regarded as a sphere at rest of radius 6400 *km* and of uniform density. How long a body will take to pass from one end of the tunnel to the other ? Show that this time is independent of the direction of the tunnel.

Let C be the middle point of a tunnel AB in the earth and O the centre of the earth (shown in section) of radius a (fig. 65), P the position of the body at any instant of time, and r, the distance of P from O.

The intensity at P due to the earth$=GM\frac{r}{a^3}$ along PO where M=mass of the earth (Art. 99, 2).

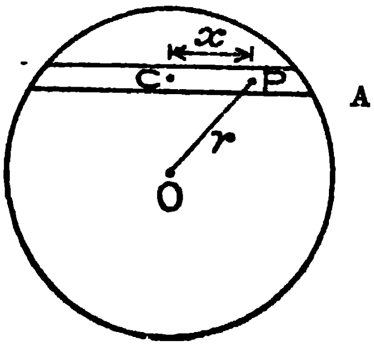

Fig. 65

The component of this towards $C=GM\frac{r}{a^3}\cos OPC$

$$=GM\frac{r}{a^3}\cdot\frac{CP}{r}=GM\frac{x}{a^3}\ (\text{where } CP=x)$$

This represents the acceleration of the body towards C as the intensity or force on unit mass is the acceleration.

The motion is therefore simple harmonic with a period T where $T=2\pi\sqrt{\frac{x}{GMx/a^3}}=2\pi\sqrt{\frac{a^3}{GM}}$ ... ... (1)

Again, the weight of unit mass on earth's surface$=1\times g=g$

This is equal to $G\frac{M\times 1}{a^2}=G\frac{M}{a^2}$

$\therefore\ g=\frac{GM}{a^2}$ or, $GM=a^2g$

Substituting for GM in (1) $T=2\pi\sqrt{\frac{a}{g}}$

$\therefore$ the time required to pass from one end of the tunnel to the other end of it$=\frac{1}{2}T=\pi\sqrt{\frac{a}{g}}$

This result is independent of the length or position of the tunnel and would be the same for tunnels bored in any direction through the earth. Taking 980 cm./sec^2 for g,

$\frac{1}{2}T=\pi\sqrt{\frac{64\times 10^7}{980}}$ sec.$=42$ minutes, approximately.

2. Given $G=6{\cdot}7\times 10^{-8}$ C.G.S. units, the radius of the earth $=6{\cdot}4\times 10^8$cm. and $g=981$ cm/sec^2. Calculate the mean density of the earth. (C. U. 1943)

$$g=G\frac{M}{R^2}=G\,\frac{4}{3}\,\frac{\pi R^3D}{R^2}=\frac{4}{3}\pi GRD$$

Where M=mass ; D, the mean density and R, the radius of the earth. g and G have the usual significances.

$$\therefore\ D=\frac{3g}{4\pi GR}=\frac{3\times 981}{4\times 3{\cdot}142\times 6{\cdot}7\times 10^{-8}\times 6{\cdot}4\times 10^8}$$

$$=\frac{3\times 981}{4\times 3{\cdot}142\times 6{\cdot}7\times 6{\cdot}4}\text{ or, } 5{\cdot}46\ gm/cc.$$

log 3= ·4771	log 4 =·6021
log 981=2·9917	log 3·142=·4972
3·4688	log 6·7 =·8261
	log 6·4 =·8062
	2·7316

log D=3·4688 − 2·7316=·7372

D=5·46 $gm/cc.$

3. The radius of the earth is $6{\cdot}37\times10^{8}$cm. its mean density $5{\cdot}5$ gm/cm^{3} and the gravitational constant $6{\cdot}66\times10^{-8}$ C.G.S. units. Calculate the earth's surface potential. (A.U. 1940)

Gravitational potential at any point on earth's surface

$$G=\frac{M}{R}=G.\frac{4}{3}\pi R^{2}D \text{ (ignoring the} -ve \text{ sign)}$$

$$=\frac{4\times6{\cdot}66\times10^{-8}\times3{\cdot}142\times(6{\cdot}37\times10^{8})^{2}\times5{\cdot}5}{3}$$

$=6{\cdot}23\times10^{11}$ ergs/gm.

4. If the moon's distance from the earth is 240,000 miles, the earth's radius is 4,000 miles g at the surface of the earth is $32{\cdot}2$ ft. per sec per sec ; find the time taken by the moon to complete one circuit of its orbit round the earth.

The gravitational attraction of the earth on the moon provides centripetal force of the moon. Hence if ρ be the density of the earth in lbs per c ft. m, the mass of the moon in lbs. T the period of revolution in secs. and G in F.P.S. units

$$\frac{G\frac{4}{3}\pi(4000\times1760\times3)^{3}\rho m}{(240{,}000\times1760\times3)^{2}}=m\times24{,}0000\times1760\times3\left(\frac{2\pi}{T}\right)^{2}$$

Art. 34...(1)

Also for the weight of a body of mass m' lbs. on earth's surface

$$\frac{G\frac{4}{3}\pi(4000\times1760\times3)^{3}\rho m'}{(4000\times1760\times3)^{2}}=m'\times32{\cdot}2 \text{ ... eqn. (1) Art. 107...(2)}$$

Dividing (1) by (2) and cancelling like factors

$$\frac{1}{(60)^{2}}=\frac{240{,}000\times1760\times3}{32{\cdot}2}\left(\frac{2\pi}{T}\right)^{2}$$

$\therefore$ Taking square root and expressing T in days

$$T=\frac{2\pi}{24}\sqrt{\frac{2\times1760}{32{\cdot}2}}=27{\cdot}37 \text{ days.}$$

5. The moon describes a circular orbit of radius $3{\cdot}8\times10^{5}$ km. about the earth in 27 days and the earth describes a circular orbit of radius $1{\cdot}5\times10^{8}$km. round the sun in 365 days. Determine the mass of the sun in terms of the mass of the earth. (Bombay 1935)

Let M_s, M_e and M_m be the masses of the sun, earth and moon. R_e the radius of earth's orbit and R_m that of the moon. V_e and V_m the speed of the earth and moon on their respective orbits.

For motion of the earth round the sun, the necessary centripetal force arises out of sun's gravitational attraction on earth. Hence

$$G\,\frac{M_sM_e}{R_e^{2}}=\frac{M_eV_e^{2}}{R_e} \quad \text{or,} \quad G\frac{M_s}{R_e}=V_e^{2} \qquad \text{...} \quad (1)$$

Similarly, for motion of the moon round the earth the necessary centripetal force arises out of earth's gravitational attraction on the moon. Hence

$$G\frac{M_e M_m}{R_m^2}=\frac{M_m V_m^2}{R_m} \quad \text{or,} \quad G\frac{M_e}{R_m}=V_m^2 \qquad \dots \quad (2)$$

Dividing (1) by (2)

$$\frac{M_s R_m}{M_e R_e}=\frac{V_e^2}{V_m^2}=\frac{\left(\frac{2\pi R_e}{365}\right)^2}{\left(\frac{2\pi R_m}{27}\right)^2}=\frac{(27R_e)^2}{(365R_m)^2}$$

$$\therefore \quad \frac{M_s}{M_e}=\frac{(27)^2R_e^3}{(365)^2R^{m3}}=\left(\frac{27}{365}\right)^2\cdot\left(\frac{1{\cdot}5\times10^8}{3{\cdot}8\times10^5}\right)^3$$

$$=3{\cdot}366\times10^5.$$

Exercise

1. State and explain Newton's Law of Gravitation, and describe an accurate method of measuring the gravitation constant.

What celestial evidence led to the formulation of the law ? Is this law universally correct ? Explain your statement. [*C. U. 1945*]

2. Find an expression for the potential at a point inside a thin uniform shell. Hence, show that the force at the point is zero. A short straight and frictionless tunnel is bored through the earth from one point of the surface to another. Assuming the earth to be a sphere of radius 4,000 miles, show that an object will travel along the tunnel under gravitational forces and will again reach the opposite face in about 42 minutes. Acceleration due to gravity is 978 cm. per sec^2, [*C. U. 1946*]

3. Explain what do you mean by gravitational potential at a point. How does it differ from other kinds of potentials with which you are familiar ? Find an expression for the gravitational potential due to a thin hollow sphere of uniform density at a point outside it. [*C. U. 1947*]

4. Explain the principle of Cavendish's experiment for finding the gravitational constant, giving a schematic diagram and deducing the formula.

Assuming that the earth is a homogeneous sphere, find the density of its material from the following data :

Radius $=6{\cdot}4\times10^8$ cm.

$G=6{\cdot}7\times10^{-8}$

$g=981$ dynes [*C. U. 1950*]

5. State Newton's law of gravitation and explain how celestial evidence led to the formation of the law.

Describe an accurate method of measuring the gravitation constant.

Is the law universally correct ? Give reasons for your answer. [*C. U. 1651*]

6. If the earth were a soild sphere of iron of radius 6·87 million metres and of density 7·86 gms. per. c.c., what would be the value of gravity at its surface, taking the gravitational constant to be $6{\cdot}58 \times 10^{-8}$ *C. G. S.* units.
[*C. U. 1952*]

7. Explain what you understand by 'Gravitation potential' and state how it differs from other kinds of potential with which you are familiar.

Prove that the gravitational potential at any external point due to a spherical shell of uniform density is the same as if the whole mass is concentrated at the centre. [*C. U. 1953*]

8. Describe and explain a laboratory method for determining the gravitational constant.

Find an expression for the attraction between a particle of mass M and an infinitely long straight wire of mass m per unit length, the distance of the particle from the wire being r. [*N. U. 1950*]

9. Describe an experiment to determine the mean density of the earth. If the gravitation constant is $6{\cdot}7 \times 10^{-8}$ *C. G. S.* units the mean radius of the earth $6{\cdot}4 \times 10^8$ cms. and the mean density is 5·5., calculate the acceleration due to gravity at the earth's surface. [*L. U. 1948*]

10. State Newton's law of gravitation and describe in detail an accurate method for determining the value of the gravitational constant. Given that $g = 980$ cm/sec^2 and $G = 6{\cdot}66 \times 10^{-8}$ *C. G. S.* units and radius of the earth $R = 6{\cdot}4 \times 10^8$ cm., calculate the average density of the earth.
[*P. U. & E. P. E. 1951*]

11. Find the gravitational potential due to a thin uniform spherical shell at any point, and hence derive expressions for the attraction on a point mass situated (1) inside the shell, (2) outside the shell.

12. Show that the weight of a body at any point in the interior of the earth varies as the distance from the centre, assuming the earth to be a homogeneous sphere. [*R. U. 1949*]

13. Write notes on determination of G by Cavendish. [*R. U. 1954*]

14. Describe and explain a laboratory for determining the gravitational constant.

Find an expression for the attraction between a particle of mass M and an infinitely long straight wire of mass m per unit length, the distance of the particle from the wire being r. [*L. U. 1950*]

15. Explain what is meant by the constant of gravitation G. [*C. U. 1959*]

Prove that the speed of projection necessary to send a body right out of the field of earth's gravitational attraction is given by $v = \sqrt{2GM/R}$ cm/sec. Where M and R are the mass and radius of the earth.

Calculate the velocity of escape if

"G" $= 6{\cdot}66 \times 10^{-8}$ dynes cm^2 gm^{-2},
$M = 5{\cdot}97 \times 10^{27}$ gm,
$R = 6{\cdot}37 \times 10^8$ cm.

CHAPTER IX

ELASTICITY

117. Rigid, Plastic and Elastic Bodies : Elastic Limit—Let a system of forces act on a body ; the forces are so arranged that they do not produce any motion (translatory or rotational) of the body as a whole. If the forces, however large they may be, fail to produce any displacement of the particles of the body, relative to one another, the body is said to be **perfectly rigid.**

Actually no body is perfectly rigid and, under the forces, there always takes place a relative displacement among the particles or parts of the body. This relative displacement among the particles deforms the body *i.e.*, alters the size or shape or both, of the body.

Materials may be sub-divided into two groups according to their behaviour, when the deforming forces are withdrawn. If the body completely retains its altered shape and size when the forces are withdrawn, the body is said to be **perfectly plastic.** If, on the other hand, the body completely recovers its original size or shape, it is called **perfectly elastic.**

No material is perfectly plastic. Every body recovers from its altered size or shape, if the deformation is not large, at least partially. Similarly, no body is perfectly elastic when subjected to large deformation. The range or limit of deformation, from which a body recovers its original size or shape, when the deforming forces are withdrawn, is called the **range of perfect elasticity** or **elastic limit.**

118. Definitions—(1) Stress—"The notion of stress, in general, is simply that of balancing internal action and reaction between two parts of a body, the force which either part exerts on the other being one aspect of *stress*."* When a body is deformed by the application of external forces, forces are called into play within the body tending to annul the deformation. The two systems of forces are evidently equal in magnitude but opposite in direction by the third law of motion. The term stress may be applied to any of the two systems of forces. When the external forces are applied uniformly on a surface, the stress is measured by the amount of force acting on unit area of the surface to which it is applied.

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*Vide Mathematical Theory of Elasticity—Love.

(2) **Strain**—The change in the relative positions of the parts of a body or the deformation which the body undergoes under the action of the applied forces is called the *strain.* According to the nature of the applied forces the strain may vary in nature.

In dealing with different types of strains we shall, however, restrict ourselves to the simplest types of strains to which bodies can be subjected. The kind of strain of a body in which any two lines in it, which were equal and parallel before straining remain equal and parallel after straining, is called a *homogeneous strain.* Thus, by a homogeneous strain a parallelogram is altered into another parallelogram, although the area or the angles of the parallelogram may be changed.

We shall consider a few of the different types of strains—

(*a*) If a wire is clamped vertically at one of its ends and loaded at the other, the strain consists of an extension in its length and is called a *longitudinal* or *tensile strain.* It is measured by the increase in length per unit length of the wire, or by its fractional change in length. The corresponding stress is sometimes called the *longitudinal* or *tensile stress.*

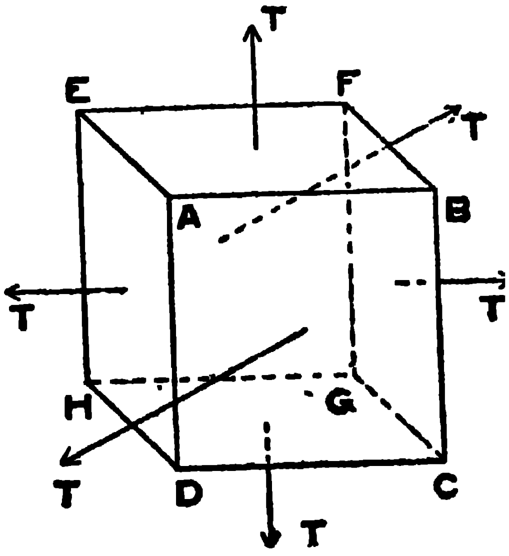

Fig. 66

(*b*) Suppose on the faces of a cube having equal sides, equal forces are applied through the middle points of the faces as shown in fig. 66. If the forces are normal to the faces to which they are applied and act outwards, their only effect is to increase the volume or bulk of the cube. The strain in this case is called *volume strain* and is measured by the change of volume per unit volume of the cube, or by its fractional change in volume. The corresponding stress is sometimes called the *volume stress.*

(*c*) Consider a cube *ABCDEFGH* (fig. 67) as above.

Let equal forces act at the middle points of the faces *AEFB* and *HDCG* *along* the respective faces as shown in the figure. These forces may be termed *tangential forces* as they act parallel to the faces to which they are applied. The effect of these forces on the cube is to alter its shape and also to rotate it as they form a couple. To prevent the rotational motion, let two other forces, each of the same magnitude as before, be applied through the middle points of the faces *AEHD* and *BCGF* *along* the faces in directions as shown in the figure. The system of forces now consists of two equal and opposite couples. Their only effect on the cube is to alter its shape, so that the face *ABCD* or its parallel, becomes a rhombus *A'B'CD* (fig. 68)* from a square.

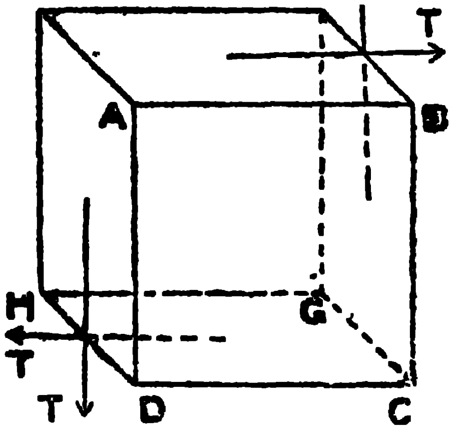

Fig. 67

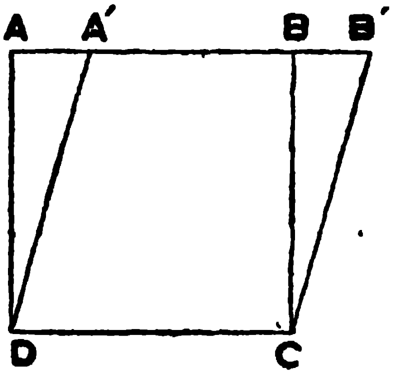

Fig. 68

This type of strain is called a *shear* and is measured by the deformation of the angle *ADC*. *i.e.*, by the angle *ADA'*, which is called the *angle of shear*. Since $\angle ADA'$, which is represented by θ, is small—a necessary assumption in the theory of elasticity, the circular measure of θ is equal to tan θ (approximately). The magnitude of the shear is thus measured by

$$\tan\theta = \frac{AA'}{AD} = \frac{\text{relative displacement between two planes}}{\text{distance of separation between them}}$$

=relative displacement between planes at unit distance apart ; also known as the *displacement gradient.*

The corresponding stress is known as the *shearing stress.*

119. Hooke's Law and the Moduli of Elasticity—Hooke, in 1678, gave the famous law of proportionality of stress and

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*See Art. 128. (*b*).

strain. The law is experimental and provides the necessary foundation for the theory of elasticity. In Hooke's words the law is *Ut tensio sic vis* : that is, the Power of any spring is in the same proportion with the Tension thereof. Using the terms 'stress and strain', the law is now stated as :—Within *elastic limit* the ratio of stress to strain is a constant. This constant is called the *modulus of elasticity* or *coefficient of elasticity*. Thus, for different natures of strain there are different moduli of elasticity as follows :—

$$\left.\begin{array}{l}\text{Modulus of longitudinal elasticity}\\ \quad\text{or Young's modulus } Y\end{array}\right\} = \frac{\text{longitudinal stress}}{\text{longitudinal strain}}$$

$$\left.\begin{array}{l}\text{Modulus of volume or bulk}\\ \quad\text{elasticity } K\end{array}\right\} = \frac{\text{volume stress}}{\text{volume strain}}$$

$$\left.\begin{array}{l}\text{Modulus of shear elasticity or}\\ \quad\text{rigidity } n\end{array}\right\} = \frac{\text{shearing stress}}{\text{shearing strain}}$$

120. Poisson's Ratio—Careful measurements show that when a body undergoes a longitudinal extension, it experiences a contraction side-ways. This lateral contraction is directly proportional to the longitudinal extension when the longitudinal extension is small. The ratio of the lateral strain to longitudinal strain is called *Poisson's ratio*. Thus, if a wire of circular section having a length L and diameter D, is subjected to a longitudinal stress which stretches it by an amount l, the *longitudinal strain* $=l/L$. If the consequent decrease in diameter amounts to d, the *lateral strain*$=d/D$.

$$\text{Poisson's ratio, } \sigma = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{\frac{d}{D}}{\frac{l}{L}}$$

If the wire be of rectangular section, the lateral strain will be measured by the decrease in length of any side of the rectangle to the original length of that side.

121. Behaviour of a stretched Wire—Fig. 69 shows the stress-strain diagram for Weld iron.* When the stress is zero, the strain is zero also and the curve passes through O. With gradual increase of stress the strain increases proportionally, so that Hooke's law of linear relationship holds from O upto the point A. The point A corresponds to the **elastic limit.** After passing A, the strain increases rather faster than that between O and A. By far the most part of this strain is permanent and the wire is said to acquire a **permanent set.** At *B*, there begins a great extension much faster than between *A* and *B* which is

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*The figure is taken from Love's Mathematical Theory of Elasticity.

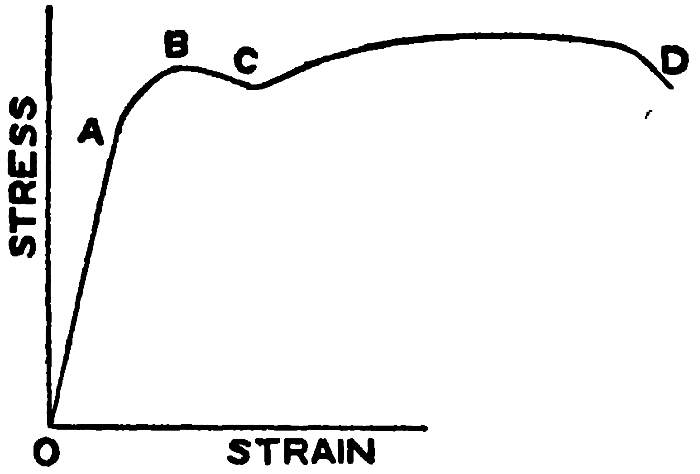

Fig. 69

shown by the bending back *BC* of the curve. The point *B* is called the **yield point.** After a further considerable increase of stress, the wire begins to thin down at some section determined apparently by accidental circumstances, where the wire ultimately breaks. The maximum stress before rupture which corresponds to the point *D* is called the **breaking stress of tenacity** of the material.

Within elastic limit, different bodies show difference in their behaviour with regard to their resumption of normal state when the deforming forces are withdrawn. A quartz thread recovers its normal length and shape almost immediately after the deforming forces are withdrawn, while a glass fibre takes several hours before it regains its normal state. The delay in recovering the normal condition of the substance, on the removal of the deforming forces, is called the **elastic after-effect.**

122. Viscosity of Metals and Elastic Fatigue—

Let two wires, one of steel and the other of zinc having the same length and diameter, be suspended vertically from rigid supports. If the free ends of the wires are loaded with two bars having the same mass and dimension, and the bars set to oscillations, it is found that the vibrations die away at very different rates. The bar attached to the steel wire vibrates for a considerable length of time but the bar connected to the zinc wire comes to rest in a short time. The decay in the vibrations of the wire is not wholly nor even mainly due to air friction, as it is the same in both the cases. The decay is mainly due to the dissipation of energy (conversion of the energy of vibration into heat) which takes place when the different layers of the wire are in relative motion. This property is called the *viscosity* of the metal from analogy with that of liquids and gases.

Another property of the metals has been discovered by Lord Kelvin. Let a pendulum be constructed by suspending a bar from a metal wire and the time of oscillations of the pendulum noted. The bar is then set to oscillations twisting the wire forwards and backwards for several days. Its time of oscillation, measured at this stage, will be found less than before. The wire behaved as if it got tired and could not keep on vibrating for a long time. This property is a change of viscosity of the metal depending upon its previous history and is known as *elastic fatigue.* If the wire is allowed to rest for some time, it recovers itself and the period of oscillation increases to its previous value.

123. Deformation of a Cube :—Relation between the Elastic Constants—

The four elastic constants Y, K, n and σ are not independent of one another but any three of them are connected by a certain relation, so that, if any two of them are found out experimentally, the other two can be deduced from the relations. To obtain the relations, let us consider a unit cube (each side of which is of unit length) of some isotropic material and deform it by suitable forces as follows :—

(a) Relation among Y, K and σ. Consider a unit cube $ABCDEFGH$ as in fig. 66. Let two equal forces, each of magnitude T, act on the cube parallel to the side AB, through the middle points of faces $AEHD$ and $BFGC$ normally outwards. The effect of these forces on the side AB or its parallel is to increase its length. The effect is similar to a wire suspended vertically from a support at one end and loaded at the other end. The weight of the load and the reaction of the support provide for the two equal and opposite forces on the wire.

If x represents the extension of the side AB or DC, we have

$$Y=\frac{\text{longitudinal stress}}{\text{longitudinal strain}}=\frac{T}{\frac{x}{1}}=\frac{T}{x}$$

Hence, the extension x, of the side AB or $DC=\frac{T}{Y}$

The two forces further make a contraction of each of the sides AD and DH on their parallels by $\sigma\frac{T}{Y}$ where σ is Poisson's ratio of the material of the cube.

Hence, the length of the sides DC, DA and DH under the forces become respectively

$$1+\frac{T}{Y},\quad 1-\sigma\frac{T}{Y}\quad \text{and}\quad 1-\sigma\frac{T}{Y}$$

If now two other forces each of magnitude T, along with the previous two, act on the cube through the middle points of the faces $AEFB$ and $DHGC$ normally outwards, the lengths of the sides DC, DA and DH under the four forces respectively become

$$1+\frac{T}{Y}-\sigma\frac{T}{Y},\quad 1-\sigma\frac{T}{Y}+\frac{T}{Y}\ \text{and}\ 1-2\sigma\frac{T}{Y}$$

If along with the four forces, two other forces, each of magnitude T, act on the cube through the middle points of the faces $ABCD$ and $EFGH$ normally outwards, the lengths of the sides DC, DA and DH, under the six forces, respectively become

$$\frac{T}{Y}-2\sigma\frac{T}{Y},\quad 1-2\sigma\frac{T}{Y}+\frac{T}{Y}\ \text{and}\ 1-2\sigma\frac{T}{Y}+\frac{T}{Y}$$

Hence, the length of each side of the cube becomes

$$1+\frac{T}{Y}(1-2\sigma).$$

The volume of the cube, under the six forces, becomes

$$\left\{1+\frac{T}{Y}(1-2\sigma)\right\}^3=1+\frac{3T}{Y}(1-2\sigma)\ \text{approximately}$$

as the quantity $\frac{T}{Y}$ is small compared to unity.

The original volume of the cube was unity, hence, the volume strain $=\frac{3T}{Y}(1-2\sigma)$* and the modulus of volume or bulk elasticity of the cube is given by

$$K=\frac{T}{\frac{3T}{Y}(1-2\sigma)}=\frac{Y}{3(1-2\sigma)}.$$

or, $Y=3K(1-2\sigma)$ ... ... (1)

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*This is evidently three times the fractional change in length or longitudinal strain $\frac{T}{Y}(1-2\sigma)$ of any side.

(b) Relation among Y, n and σ. Next, suppose the unit cube is under the action of four forces (as in fig. 67) each of magnitude T. The effect of the forces is only to alter its shape. The alteration will be brought about by the elongation of the diagonal plane through DB and contraction of the diagonal plane through AC as is explained below.

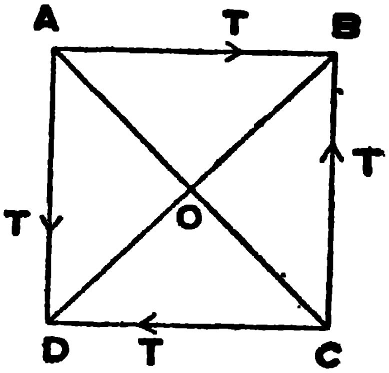

Fig. 70

Consider the face $ABCD$ of the cube Let the two diagonals AC and DB intersect at O. (fig. 70).

The force T, which is represented by AB, is equivalent to two forces represented in direction and magnitude by AO and OB. Since the length AB is unity, each of the lengths AO, BO, CO and DO amounts to $1/\sqrt{2}$. Hence, the force T, represented by AB is equivalent to (1) $T/\sqrt{2}$ along AO and (2) $T/\sqrt{2}$ along OB. Similarly, the force T represented by CB is equivalent to (1) $T/\sqrt{2}$ along CO and (2) $T/\sqrt{2}$ along OB. In the same way the force T represented by AD is equivalent to (1) $T/\sqrt{2}$ along AO and (2) $T/\sqrt{2}$ along OD and the force T represented by CD is equivalent to (1) $T/\sqrt{2}$ along CO and (2) $T/\sqrt{2}$ along OD. On considering all the forces along DB denoted by the components (2) of the four forces, we see that they produce an extension of the diagonal DB, the equivalent extensional force being $\frac{T}{\sqrt{2}}+\frac{T}{\sqrt{2}}$ or $\sqrt{2}T$. This force along the diagonal DB acts across the diagonal plane $ACGE$ (fig. 67) through AC, the sides of which are $\sqrt{2}$ and 1 and hence across an area $\sqrt{2}$.

$$\therefore \text{ the longitudinal stress along } DB=\frac{\sqrt{2}T}{\sqrt{2}}=T$$

If x_1 is the extension of the diagonal DB, we evidently have,

$$Y=\frac{T}{\frac{x_1}{\sqrt{2}}}=\frac{\sqrt{2}T}{x_1}. \quad \text{Hence } x_1=\frac{\sqrt{2}T}{Y}.$$

Another effect of this extensional force along DB is to contract the diagonal AC by y_1 where $y_1=\sigma\frac{\sqrt{2}T}{Y}$.

Similarly, on considering the forces along *AC*, denoted by the components (1) of the four forces we see that they form a compressive force along *AC*, the equivalent magnitude of which is $\sqrt{2}T$, acting across the diagonal plane *DBFH* (fig. 67) through *DB*. Thus, proceeding in the same way as before this force contracts the diagonal *AC* by y_2, where $y_2=\frac{\sqrt{2}T}{Y}$ and extends the diagonal *BD* by x_2, where $x_2=\sigma\frac{\sqrt{2}T}{Y}$.

$\therefore$ the total extension of the diagonal *DB*

$$=x_1+x_2=\frac{\sqrt{2}T}{Y}(1+\sigma)^*$$

and the total compression of the diagonal *AC*

$$=y_1+y_2=\frac{\sqrt{2}T}{Y}(1+\sigma)^*$$

Because of these changes in length of the diagonals *DB* and *AC*, the face *ABCD* becomes a rhombus from a square (as shown in fig. 68).

The fractional change in lengths or the longitudinal strain of the diagonal *DB* cr *AC*

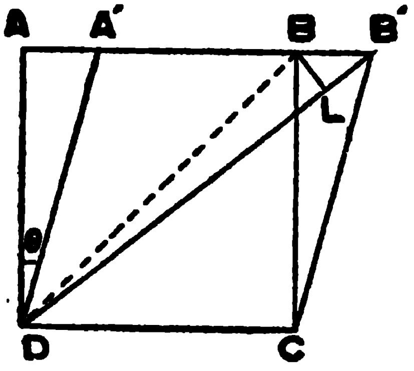

Fig. 71

$$=\frac{\frac{\sqrt{2}T}{Y}(1+\sigma)}{\sqrt{2}}$$

$$=\frac{T}{Y}(1+\sigma) \qquad (2)$$

The shearing strain θ can be proved to be twice the longitudinal strain of the either diagonal. Fig. 71 shows the deformed shape *A'B'CD* of the original face *ABCD*; *A'C* and *DB'* are the compressed and extended diagonals. Drop a normal *BL* on *DB'*.

The longitudinal strain of the diagonal *DB*

$$=\frac{LB'}{DL}=\frac{LB'}{DB} \text{ (since } DB=DL \text{ nearly)}$$

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*For this, a shear is regarded as a combination of an extension together with an equal compression perpendicular to the extension both of which are at 45° to the direction of the shear (AB).

$$=\frac{BB'\cos BB'L}{\sqrt{2}BC}=\frac{\frac{BB'}{\sqrt{2}}}{\sqrt{2}BC}\left(\text{as } \angle BB'L=\frac{\pi}{4}\text{ nearly}\right)$$

$$=\frac{BB'}{2BC}=\frac{\theta}{2}\text{ (Art. 118, 2 } c).$$

Thus, the shearing strain θ of the cube$=2\times$the longitudinal strain of the either diagonal DB or AC and is given by

$$\theta=2\frac{T}{Y}(1+\sigma) \qquad \ldots \qquad \text{(from equation 2)}$$

The modulus of rigidity n of the cube is thus given by

$$n=\frac{T}{\theta}=\frac{T}{2\frac{T}{Y}(1+\sigma)}=\frac{Y}{2(1+\sigma)}$$

or, $Y=2n(1+\sigma)$ .... (3)

(*c*) **Relation among the Elastic Moduli and the Limits of σ.**

Eliminating σ from equations (1) and (3),

we have $Y=\dfrac{9nK}{3K+n}$ ... ... (4)

and eliminating Y from the same two equations

$$\sigma=\frac{3K-2n}{6K+2n} \qquad \ldots \qquad \ldots \qquad (5)$$

Equation (4) gives the relation among the three moduli of elasticity and equation (5) gives the value of σ in terms of K and n, σ is also obtained in terms of Y and K in equation (1) and in terms of Y and n in equation (3).

It can be shown that the value of σ, for any material, lies between $\frac{1}{2}$ and -1.

Comparing equations (1) and (3)

$$3K(1-2\sigma)=2n(1+\sigma) \qquad \ldots \qquad (6)$$

Now, n and K are positive quantities. If σ has a positive value, the right side of (6) is a positive quantity ; for the left side of (6) to be positive

$$1-2\sigma>0 \text{ or, } \frac{1}{2}>\sigma$$

If σ has a negative value, the left side of (6) is a positive quantity ; for the right side of (6) to be positive,

$$1+\sigma>0 \text{ or, } \sigma>-1$$

Hence, σ lies between $\frac{1}{2}$ and -1 or, σ satisfies the inequality

$$\frac{1}{2}>\sigma>-1.$$

124. Elasticity possessed by different kinds of Matter—A solid possesses both shear elasticity and bulk elasticity. If the *shape* of a solid is altered by applying a shearing stress and if it completely regains its former shape on the removal of the stress, it is called a **perfect solid.** If the shearing stress does not surpass a certain value depending on the material of the solid, many solid bodies satisfy this condition.

A fluid possesses bulk elasticity but has no shear elasticity. A fluid thus cannot permanently* resist a tangential stress. A fluid which cnnnot offer a tangential stress whether at rest or in relative motion, is called a **perfect fluid.** But in practice, a relative motion, among the different layers of a fluid is accompanied by tangential forces tending to prevent the relative motion and these tangential forces persist as long as the relative motion lasts. This property of the fluid is known as **viscosity** and may be regarded as a transient type of the rigidity. It will be treated further in the Chapter on Viscosity.

A liquid which is completely incompressible is usually described as **a perfect liquid.** In practice, no liquid is completely incompressible, although the change in volume for moderate increase of pressure is very small. Gases, on the other hand, respond easily to changes of pressure. Thus, the modulus of bulk elasticity of a liquid is very large compared to that of a gas. The reciprocal of the bulk modulus is called the **compressibility.**

A gas, the volume of a certain mass of which is in reciprocal proportion to its pressure at constant temperature, in other words, a gas which obeys Boyle's law is called a **perfect gas.**

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*See Chapter IX.

125. Determination of Young's Modulus—A simple way of determining Young's modulus for a material in the form of a long wire, is shown in fig. 72. Two long wires *AB* and *CD* of the same material are suspended vertically close to each other from the same firm support.

Both the wires carry scale-pans at their lower ends. The wire *CD* carries a scale *S* graduated in millimetres and the wire *AB* carries a vernier *V* placed alongside the scale. Each wire is loaded sufficiently to straighten it. The weight on the scale-pan attached to wire *AB* is then increased by equal steps and then decreased in same steps..From the mean of the readings of the vernier the vertical depression of the zero point of the vernier, produced by a weight, is obtained. The weight must not be too large to stretch the wire beyond elastic limit of the wire. The extension of the wire *AB* is then plotted against the load applied. The curve will be a straight line. Measuring the cross-section and the length of the wire, the stress and the longitudinal strain produced by a particular weight are calculated and Young's modulus of the material of the wire found out.

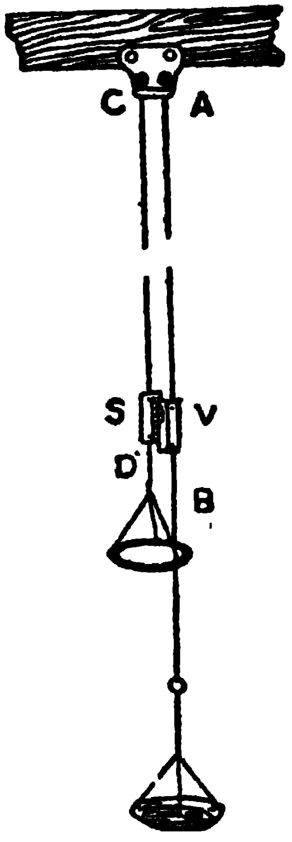

Fig. 72

An improvement in measuring the stretching of the wire *AB* is due to Mr. G.F.C. Searle. Two brass frames *EF* and *GH* (fig. 73) hang from the lower ends of the wires *CD* and *AB* and support the two ends of a straight spirit level *L*. One end of the spirit level is pivotted to the frame *EF* and the other end rests on a micrometer screw *S* working in the frame *GH*. The brass frames are connected together by links *K*, *K'* to prevent the frames from twisting the wires relative to each other but they freely allow the vertical relative motion of the frames.

From the lower end of the frame *EF*, hangs a weight to straighten the wire *CD*. A sufficient weight is placed in the scale-pan attached to the frame *GH*, to keep the wire *AB* straight and the bubble of the spirit level is brought to the centre. The weights in the scale-pan are increased by equal steps and the bubble of the spirit level adjusted to the centre by turning the screw *S*. The weights are then decreased in the same steps. The weights must not be too large to stretch the wire beyond its elastic limit.

The stretching of the wire AB is thus found out from the mean of the readings of the micrometer screw along a linear scale and

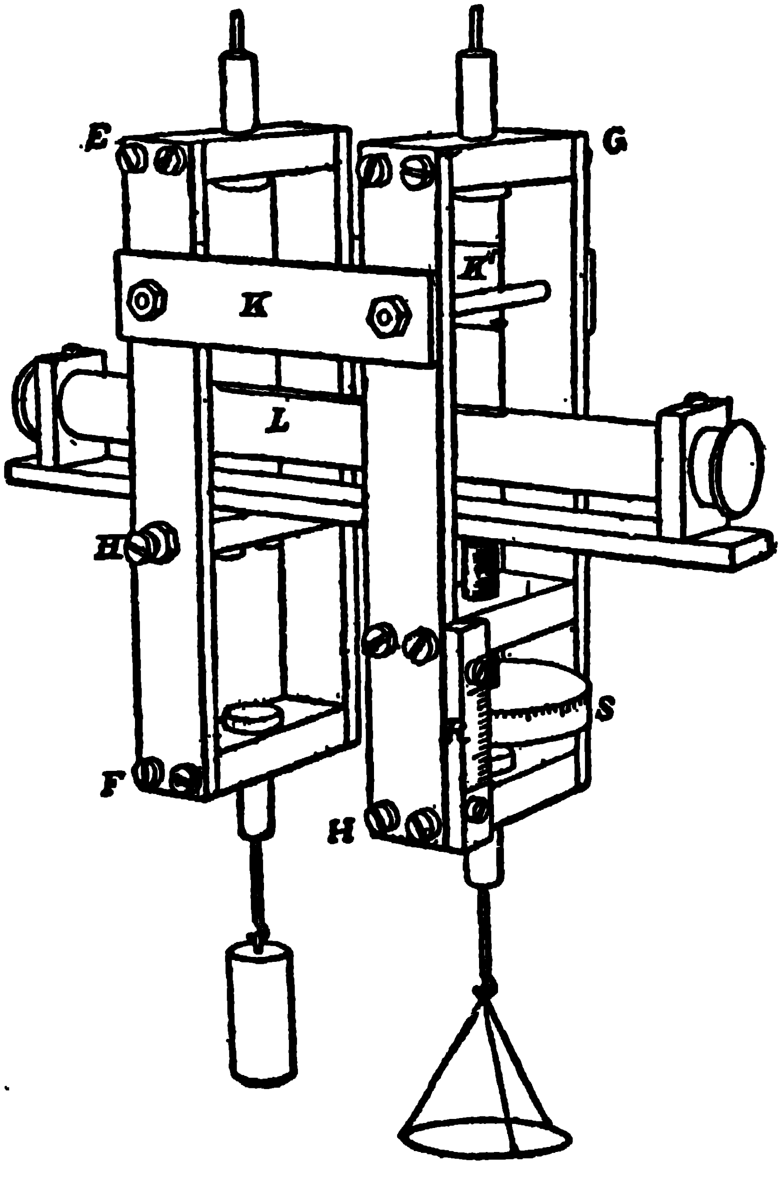

Fig. 73

is plotted against the load applied. The curve is a straight line from which Young's modulus of the wire is calculated.

The suspension of the two wires from, the same support prevents any relative displacement between the wires due to yielding of the support, if any, and both the wires are taken of the same material to avoid the relative displacement due to any change of temperature which may take place during the experiment.

126. Torsion of a Cylinder—Let a cylinder of length l and radius of cross-section r be fixed at the upper end and twisted by the application of a couple at the lower end (fig. 74). This is an example of a pure shear as the length or the radius of the cylinder does not undergo any change.

Consider a coaxial cylindrical tube within the solid cylinder and two parallel radii O_1A and O_2B at the ends. The tube is of length l and radius x which is equal to O_1A or O_2B and thickness dx and forms a part of the cylinder. Due to the couple applied at the lower end, the line AB of the inner surface of the tube is displaced to the position AC. The angle BAC or ϕ is the *shear* and the angle BO_2C or θ is the *twist* produced. Evidently

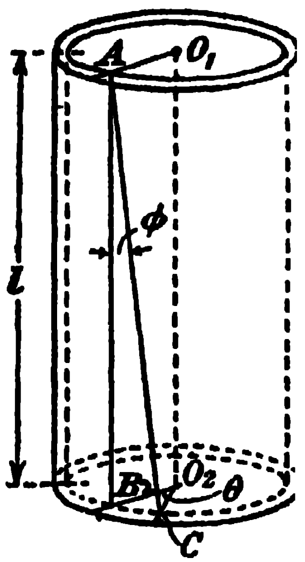

Fig. 74

$$l\phi = x\theta \quad \dots \quad (1)$$

The tangential stress p acting on the base of the tube is given by

$$p = n\phi \quad \dots \quad (2)$$

where n=modulus of shear elasticity.

The tangential force acting on the base of the tube

$$= p.\ 2\pi x dx = 2\pi n\phi x dx \text{ from (2)}$$

$$= \frac{2\pi n\theta}{l} x^2 dx \text{...from (1)}$$

The moment of this force about the axis of the tube

$$= \frac{2\pi n\theta}{l} x^3 dx$$

The moment of the couple applied to all the elementary coaxial cylindrical tubes forming the solid cylinder to produce the twist θ

$$= \int_{0} \frac{2\pi n\theta}{l} x^3 dx = \frac{\pi n r^4 \theta}{2l}$$

If the cylinder is hollow of internal radius r and external radius R, the moment of the couple needed to produce the twist θ

$$=\int^{R}\frac{2\pi n\theta}{l}x^{3}dx=\frac{\pi n}{2l}(R^{4}-r^{4})\theta$$

127. Determination of Rigidity Modulus.—The determination of rigidity modulus is made—(*a*) from the period of oscillation of *torsion pendulum.* In its simplest form (fig. 75). the torsion pendulum consists of a long uniform wire made of the material under examination and suspended vertically from a rigid support. At the lower end, the wire carries a massive cylinder of brass, fixed coaxially to it. To make an experiment, a thin vertical mark is made on the surface of the cylinder, the image of which is brought to coincide with the vertical line of the cross-wire of a telescope focussed on the cylinder. The cylinder is turned through a small angle whereby the wire is twisted. The twist in the wire alters its shape and by virtue of shear elasticity, the wire tends to untwist itself. While untwisting, the wire gains kinetic energy of rotation at the cost of potential shear energy in the twisted condition and is set to torsional oscillations. The period of oscillation is measured by observing the turnings of the mark on the surface of the cylinder through the telescope in the usual way,

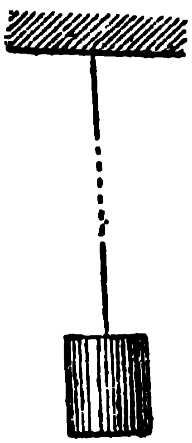

Fig. 75

The angular acceleration of the oscillating cylinder is given by

$$\frac{d^{2}\theta}{dt^{2}}=\frac{\text{moment of the shearing (turning) couple}}{\text{moment of inertia of the vibrating system about the axis of rotation}}$$

$$=\frac{\pi n r^{4}\theta}{2lI}\qquad\text{(from Art. 126)}$$

where n is the modulus of shear elasticity of the material of the wire ; l, its length ; r, its radius and I, the moment of inertia of the cylinder about its axis.

The angular acceleration is thus proportional to the angular displacement and is oppositely directed to it. The cylinder, therefore, executes angular simple harmonic motion. The period of oscillation, T is given by

$$T=2\pi\sqrt{\frac{2lI}{\pi n r^4}} \qquad \text{(Art. 47)}$$

$$\therefore \quad n=\frac{8\pi lI}{T^2r^4}$$

This method is known as the *dynamical method*, as observations are made on the material under examination while it is in a state of motion.

(*b*) Another method, in which the observations are made while the wire is at rest, is known as the *statical method* to distinguish it from the previous one.

The apparatus, (fig. 76) due to Barton, used in the statical method consists of a wire of the material to be examined, comparatively thick and short rather than that used in the dynamical method. The wire, is suspended vertically from a rigid support and carries a massive cylinder. at its lower end, firmly attached to it which serves to keep the wire taut. A piece of string is attached at its middle to a small pin on the cylinder and the two portions of the string turn round the cylinder in the same direction without overlapping. The strings, after two or three turns, leave the cylinder tangentially at the opposite extremities of a diameter of the cylinder. The strings then pass horizontally over two smooth pulleys fixed on the support and carry two scale-pans of equal weight at their ends. At the centre of one end of the cylinder a long pointer is attached which moves on a fixed circular scale graduated in degrees.

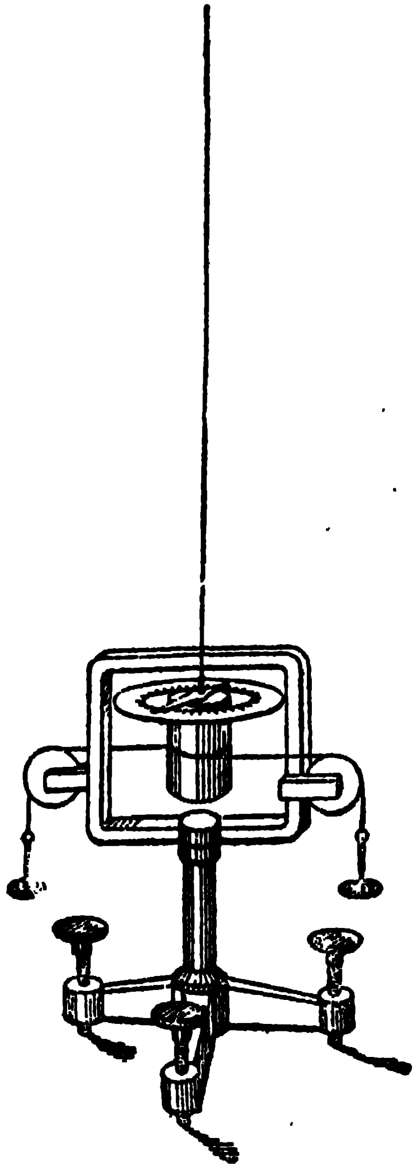
Fig. 76

To make an experiment, equal weights are placed on the scale-pans. These weights provide a couple twisting the wire. The deformation in the shape of the wire exerts a restoring couple. After a certain twist of the wire, measured from the shift of the pointer on the graduated scale, equilibrium is established. Weights on the two scale-pans are increased in equal steps and the corresponding twist θ is measured ; the weights are then decreased in the same equal steps and the mean values of θ for the two sets of readings are plotted against the corresponding weights. The graph will be a straight line from which the twist θ for a particular load mg is found out. If the diameter of the cylinder, round which the string is wound is d. the moment of the applied couple$=d.mg$. Hence, from Art. 126.

$$\frac{\pi n r^4 \theta}{2l}=dmg \quad \text{or,} \quad n=\frac{2ldmg}{\pi r^4 \theta}$$

where l and r are the length and radius of the wire ; θ, the circular measure of the angle of twist measured from the shift of the pointer along the circular scale and n, the modulus of shear elasticity of the wire.

It will be noticed that in both the methods n depends upon the fourth power of the radius of the wire. Therefore, great care should be taken in measuring the radius. The measurement should be taken at some six places on the wire and at each place twice in rectangular direction. The mean of the twelve values thus obtained, affords a satisfactory measure of the radius.

128. Determination of Poisson's Ratio and Bulk Modulus of a Solid—Finding Young's modulus and the rigidity modulus of a solid experimentally, its bulk modulus is determined from relation (4) and Poisson's ratio from relation (3) of Art. 123. The bulk modulus of a solid can, however, be determined by independent experiment in which a short solid cylinder of large diameter is enclosed in a strong glass vessel which is filled with water. The water is subjected to high pressure. By observing two marks on the cylinder through two microscopes before and after the application of the pressure, the decrease in length per unit length of the cylinder produced by a uniform compressive stress is measured. The decrease in volume per unit volume or the volume strain will be three times the decrease in length per unit length Art. 123, foot-note). The bulk modulus is given by the ratio of the pressure applied (which is measured by a manometer) to the volume strain.

127. The Bending of Bars : Cantilever—Let a bar be clamped horizontally at one end and loaded at the other. By the action of the applied load the natural shape of the bar will be altered but it will recover from the strain on withdrawal of the load provided no part of it has been strained beyond the elastic limit. A system which is clamped and loaded in the above manner is called a *cantilever*.

Fig. 77 shows the section of a bar by a plane perpendicular to the breadth of the bar which is clamped horizontally at the end A and loaded at the end B. Let l be the length; b, the breadth and d, the thickness of the bar. The effect produced by the load is that the upper half will be stretched and the lower half compressed along the length of the bar. The middle layer will neither be stretched nor compressed but will simply be curved: this layer is called the *neutral layer*. The stretching or compression of a layer will increase according as its distance from the neutral layer increases. The system of elastic forces which will be called into play within the bar tending to counteract the extension or compression, across any section POQ will gradually increase both above and below the neutral layer as shown qualitatively in direction and magnitude by arrow-heads drawn across this section. These elastic reactions thus produce a system of couples whose resultant may be called the *moment of resistance* or the *internal bending* moment. The applied force F acting vertically downwards at B with its reaction F acting vertically upwards across the section POQ (not shown in the figure) form a couple the moment of which tends to bend the bar and is thus called the *external bending moment*. A section of the neutral layer by the plane of this couple is called the *neutral axis*.

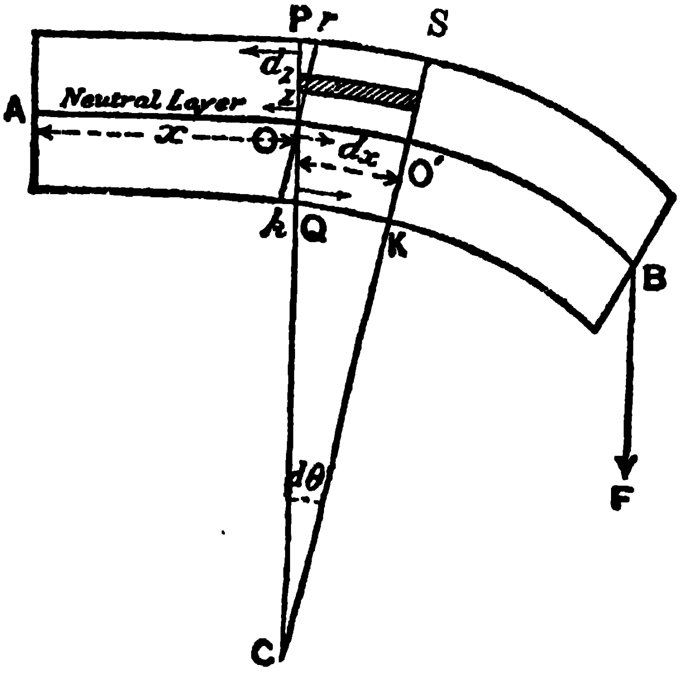

Fig. 77.

To calculate the moment of resistance, consider an element of length dx of the bar between two points O and O' on the neutral axis, the element being at a distance x from the end A. The normals POQ and $SO'K$ to the neutral layer at the two extremities of the element which were parallel to each other in the unstrained condition of the bar meet each other at C and are inclined at an angle $d\theta$ (say) to each other when the bar is strained. If through O, a line rOk be drawn parallel to $SO'K$, the distance between POQ and rOk at any height above or below the neutral layer shows the extension or compression of a length dx at that height.

Let us imagine a strip (shaded) of length dx, breadth b and thickness dz at a height z from the neutral layer. The extension of this strip is $zd\theta$ as the angle POr=angle $PCS=d\theta$. Hence, if f represents the elastic reaction of the strip

$\frac{f}{bdz}=Y\frac{zd\theta}{dx}$ (where Y=Young's modulus of the bar and $b.dz$ =cross section of the strip).

Also from similar triangles

$\frac{dx+zd\theta}{dx}=\frac{R+z}{R}$ (where $R=OC$, the radius of curvature of neutral layer at O.)

$$\text{or, } \frac{zd\theta}{dx}=\frac{z}{R}$$

$$\therefore \frac{f}{bdz}=Y\frac{z}{R} \text{ or, } f=\frac{Y}{R}bzdz$$

The moment of this elastic reaction about neutral axis

$$=fz=\frac{Y}{R}bz^2dz$$

The moment of elastic reaction on the whole cross-section

$$=\Sigma fz=\frac{Y}{R}\Sigma bz^2dz$$

Σbz^2dz is called the *moment of inertia of the cross-section* of the bar about the neutral axis or the *geometrical moment of inertia.* Representing this by i, the moment of elastic reaction on the cross-section of the bar

$$=\frac{Yi}{R} \quad \dots \quad \dots \quad \dots \quad (1)$$

The radius of curvature R at a point (x, y) is given by

$$R=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}$$

For small bending. $\frac{dy}{dx}$ is small, therefore, $\frac{1}{R}=\frac{d^2y}{dx^2}$...(2) where y is the displacement of the neutral axis at a distance x from the fixed point A. From (1) and (2) the moment of elastic reaction or moment of resistance$=Yi\frac{d^2y}{dx^2}$. For equilibrium of the portion OB of the bar this must be equal to the external bending moment $F(l-x)$.

Hence, for equilibrium of the portion OB of the bar

$$Yi\frac{d^2y}{dx^2}=F(l-x)$$

Integrating $\int Yi\frac{d^2y}{dx^2}dx=\int F(l-x)dx+c$

or, $Yi\frac{dy}{dx}=F\left(lx-\frac{x^2}{2}\right)+c$ where c is an integration constant.

To evaluate c, $\frac{dy}{dx}=0$ at $x=0$ as the bar is horizontal at A.

Hence $c=0$

$$\therefore \quad Yi\frac{dy}{dx}=F\left(lx-\frac{x^2}{2}\right)$$

Integrating again $\int Yi\frac{dy}{dx}dx=\int F\left(lx-\frac{x^2}{2}\right)dx+c'$

or $Yiy=F\left(\frac{lx^2}{2}-\frac{x^3}{6}\right)+c'$ where c' is an integration constant

To evaluate c', $y=0$ at $x=0$ as the end A is undisplaced.

Hence $c'=0$

$$\therefore \quad Yiy=F\left(\frac{lx^2}{2}-\frac{x^3}{6}\right) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

The value of i which is not the true moment of inertia but the moment of inertia of the cross-section or *geometrical moment of inertia* of the bar (of rectangular section) about the neutral axis can be calculated as follows. The cross-section of the strip is bdz and its moment of inertia about neutral axis is $bdz.z^2$. The moment of inertia of the whole cross-section about the axis

$$=2\int_0^{d/2} bz^2 dz=\frac{1}{12}bd^3$$

Putting this value of i in equation (2)

$$\frac{Ybd^3}{12}y=F\left(\frac{lx^2}{2}-\frac{x^3}{6}\right) \qquad \ldots \qquad (3)$$

Equation (3) is the equation to neutral axis of the bar

Corresponding to $x=l$ (at the free-end), the depression y is a maximum $=\delta$ (say)

Putting these values in (3)

$$\frac{Ybd^3}{12}\delta=\frac{Fl^3}{3} \quad \text{or,} \quad \delta=\frac{4Fl^3}{Ybd^3} \qquad \ldots \qquad (4)$$

The value of i for a rod (of circular section) can be calculated as follows. If an annulus concentric with the cross-section having radius x and width dx is considered, its area is $2\pi xdx$ and its moment of inertia about an axis through the centre of cross-section perpendicular to its plane is $2\pi xdx.x^2$. The moment of inertia of the whole cross-section about the axis

$$=\int_0^r 2\pi x^3 dx=\tfrac{1}{2}\pi r^4 \text{ (where } r=\text{radius of cross-section)}$$

The moment of inertia about the neutral axis which is a diameter of the cross-section$=\frac{1}{4}\pi r^4$ (Art 73, 4 cor *ii*).

Putting this value of i, equation (2) becomes

$$\frac{\pi Yr4}{4}y=F\left(\frac{lx^2}{2}-\frac{x^3}{6}\right) \qquad \ldots \qquad (5)$$

$$\text{The depression } \delta \text{ (at the free end)}=\frac{4Fl^3}{3\pi Yr^4} \qquad \ldots \qquad (6)$$

The relation (4) or (6) admits of an experimental determination of Y from an observation of δ, the displacement at the furthest end B of the bar produced by a force F.

130. Bar supported at both ends and loaded at the middle —A rectangular bar AB of length l is supported on knife-edges at its ends and a load F is suspended from the middle point C of it. (fig. 78) Due to the downward force F at the middle, forces are exerted at the two entremities due to the reaction of the knife-edges. These forces are $F/2$ at each end and act vertically upwards.

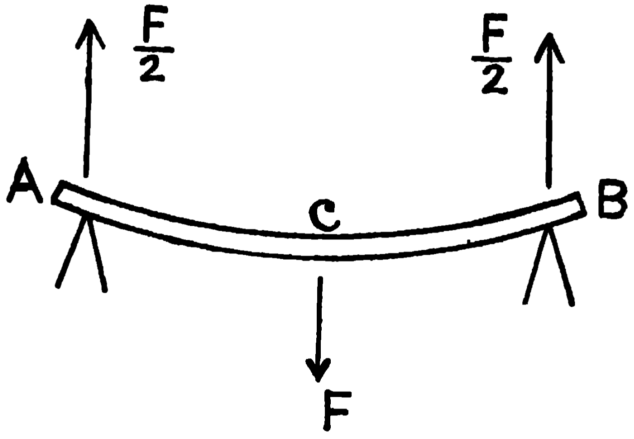

Fig. 78

Considering each half AC or BC of the bar it presents a cantilever as the bar is horizontal at the middle point C and either end is displaced by a force $F/2$. Hence, the depression of C relative to A will be that of a cantilever of length $l/2$ kept horizontally clamped at one end and depressed by a force $F/2$ and the other.end of it. Thus, in the expression for (4) of Art. 129 for maximum depression of a cantilever if we put $F/2$ for F and $l/2$ for l, other quantities remaining the same, we shall get the displacement δ of C relative to A or B. Thus.

$$\text{for a bar, } \delta = \frac{4\frac{F}{2}\left(\frac{l}{2}\right)^3}{Ybd^3} = \frac{Fl^3}{4Ybd^3} \text{ and}$$

$$\text{for a rod, } \delta = \frac{4\frac{F}{2}\left(\frac{l}{2}\right)^3}{3\pi r^4 Y} = \frac{Fl^3}{12\pi r^4 Y} \text{ from equation (6) of Art 129}$$

131. Vibration of Loaded Bars—The depression δ of the bar in any of the above two cases is proportional to the weight suspended. Let a mass M suspended in the above way produce depression δ in any of the above two cases. Then,

$\delta = KMg$ where $K = \frac{4l^3}{Ybd^3}$ for a cantilever and $K = \frac{l^3}{4Ybd^3}$ for a bar supported at ends and loaded at the middle.

$$\therefore \quad Mg = \delta/K$$

The system is depressed further by an increased force and released. Let at any time t, x be the further depression. The upward elastic reaction is $(\delta+x)/K$. Therefore, the resultant upward force

$$=\frac{\delta+x}{K}-Mg=\frac{\delta+x}{K}-\frac{\delta}{K}=\frac{x}{K}$$

The acceleration of motion is given by

$$\frac{d^2x}{dt^2}=\frac{x}{KM}$$

The acceleration is thus proportional to the displacement and directed oppositely to it. The motion is, therefore, simple harmonic its period is given by $T=2\pi\sqrt{KM}$ (Art. 47) where the value of K is given previously to suit particular cases.

132. Potential Energy of a Strained Body—The potential energy of a body strained in any manner is due to the work done in producing the strain.

(1) Potential energy of strained wire (longitudinal strain) —Let a wire of length l be stretched by an amount x due to the application of a tensile force F. The work done in increasing the extension by $dx=F.dx$

Hence, the work done in producing the extension x

$$=\int_0^x F dx \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Again $\frac{F}{A}=Y\frac{x}{l}$ $\quad\therefore\quad F=AY\frac{x}{l}$ (Y=Young's modulus) (A=area of cross-section)

$\therefore$ Work done in producing the extension x from (1)

$$=\int_0^x AY\frac{x}{l}dx=\frac{1}{2}AY\frac{x^2}{l}$$

This work remains as potential energy in the strained wire.

The potential energy per unit volume of the wire

$$=\frac{1}{2}AY\frac{x^2}{l}\cdot\frac{1}{Al}=\frac{1}{2}Y\frac{x^2}{l^2}\ldots(2)\ (Al=\text{volume of the wire})$$

Also the stress$=\frac{F}{A}=\frac{Yx}{l}$ and the (longitudinal) strain$=\frac{x}{l}$

Hence, the potential energy per unit volume of the wire from (2)

$$=\frac{1}{2}\frac{Yx}{l}\cdot\frac{x}{l}=\frac{1}{2}\,stress\times strain.$$

(2) Potential energy of shear—Consider a cube each side of which is of length l. A tangential force F' is applied to one of its faces the opposite face being kept fixed. Let the relative displacement between the faces be x. The work done in increasing the displacement by $dx=F'dx$

Hence, the work done in producing the displacement x

$$=\int_0^x F'dx \qquad \ldots \qquad \ldots \qquad (1)$$

Again, $\frac{F}{A}=n\frac{x}{l}$ (n=modulus of rigidity) (A=area of the face)

or, $F=An\frac{x}{l}$

∴ The work done in producing the displacement x from (1)

$$=\int_0^x An\frac{x}{l}dx=\frac{1}{2}An\frac{x^2}{l}$$

This work remains as potential energy in the strained cube. The potential energy per unit volume of the cube

$$=\frac{1}{2}An\frac{x^2}{l}\cdot\frac{1}{Al}=\frac{1}{2}\frac{nx}{l}\cdot\frac{x}{l}\ldots(2)\ (Al=\text{volumo of the cube})$$

Also the stress $\frac{F}{A}=n\frac{x}{l}$ and the (shearing) strain$=\frac{x}{l}$

Hence, the potential energy per unit volume of the cube from (2)

$$=\tfrac{1}{2}\,stress\times strain.$$

(3) Potential energy of a compressed gas

The work done in compressing a gas is the source of its potential energy. The compression can be made under two conditions —(*a*) isothermal and (*b*) adiabatic.

(*a*) *Work done in Isothermal Compression*

Let a gramm-molecule of gas be compressed from a volume v_1 to v_2 isothermally. To caculate the work done, we observe that the work done for compression through dv by an increase of pressure p is $-pdv$ where the $-ve$ sign indicates that for an increase of pressure p, the decrease of volume is dv. Hence, work done in compression from v_1 to v_2

$$=\int_{v_1}^{v_2} -pdv = \int_{v_1}^{v_2} -\frac{RT}{v}dv \text{ (as for a perfect gas } pv=RT\text{)}$$

where T is the absolute temp. of the gas and R. the universal gas constant. Therefore, the work done during compression

$$= -RT \log_e \frac{v_2}{v_1} = RT \log_e \frac{v_1}{v_2}$$

If instead of a *gm. mol.*, a certain mass m of the gas is compressed, R will be replaced by Rm/M where M is the molecular weight of the gas.

(*b*) *Work done in Adiabatic Compression*

Let a certain mass of gas be compressed adiabatically from a volume v_1 to a volume v_2 ; its pressure changing from p_1 to p_2. The work done during compression as before

$$=\int_{v_1}^{v_2} -pdv$$

Since, for a perfect gas the adiabatic condition is given by

$$pv^{\gamma}=k \text{ (a const.)}, p=\frac{k}{v^{\gamma}}$$

$$\therefore \text{ Work done}=\int_{v_1}^{v_2} -\frac{k}{v^{\gamma}}dv=\frac{k}{\gamma-1}\left(v_2^{-\gamma+1}-v_1^{-\gamma+1}\right)$$

Replacing k by $p_2v_2^{\gamma}$ and $p_1v_1^{\gamma}$ respectively as multipliers of the two terms within brackets, the work done

$$=\frac{p_2v_2-p_1v_1}{\gamma-1}.$$

133. Determination of the Bulk Modulus of a Liquid—Experimental measurement of the bulk modulus of a liquid is beset with the difficulty that when a liquid, enclosed in a vessel, is subjected to a high pressure, the containing vessel yields. To eliminate this, Regnault in 1847 enclosed the liquid under examination in a thin walled glass vessel *A* (fig. 79) called piezometer having the shape of a thermometer. The level of the liquid stood in the narrow glass tube *E*, which the vessel *A* was provided with. The tube *E* had graduations from which the volume of the liquid enclosed could be obtained. The vessel *A* was enclosed in a larger vessel containing water and having a stem provided with a stop-cock *D*. This stem was connected to the tube *E* between two stopcocks *C* and *T* with which the tube *E* was provided. The vessels were kept immersed in a bigger vessel *B* containing water for keeping the temperature constant.

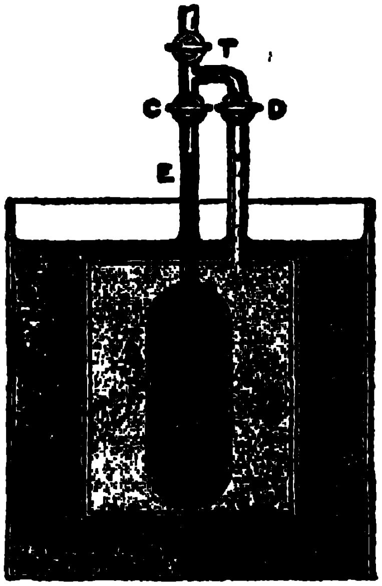

Fig. 79

By connecting the end of the tube *E* to a compression pump and opening all the stop-cocks, the same high pressure was applied to the liquid in *A* and to the outer surface of the vessel *A*. The drop of liquid meniscus in *E* was observed and the pressure was noted by manometer.

It would appear at first sight that because the liquid in *A* as well as the outer surface of the vessel *A* was subjected to the same pressure, there will be no change in the internal volume of of the vessel *A*. But the walls of the vessel *A* being subjected to the same pressure both internally and externally will diminish in volume which will alter the internal volume of the vessel. The effect of contraction in volume of the walls on the internal volume of the vessel can be studied by comparing it with a vessel containing a liquid which is heated. Due to heat, the walls of the vessel expand and this expansion produces an increase in the internal volume of the vessel and in consequence the meniscus of liquid in the vessel falls.

The effect of contraction in volume of the walls in the present case is thus to decrease the internal volume of the vessel.* This decrease in volume can be calculated by supposing the inner hollow space of the vessel *A* to be filled up with the material of the vessel, which is subjected to the same pressure as applied to the liquid. The diminution of the internal volume of the vessel *A* in the experiment will be the same as the diminution in volume of the solid core that fills the hollow space within A.

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* For a mathematical study of this see Newman and Searle's Properties of matter Art. 68.

Therefore, if the apparent decrease in volume of the liquid as observed by the drop of the meniscus be v_1 and if v_2 be the decrease in volume of the solid core under the same pressure, the real decrease in volume v of the liquid is given by

$$v=v_1+v_2 \qquad \dots \qquad \dots \qquad \dots \qquad (1)$$

Let P be the applied pressure and V the volume of the vessel A.

From (1), $\frac{v}{V}=\frac{v_1}{V}+\frac{v_2}{V}$

or, $\frac{v_1}{V}=\frac{v}{V}-\frac{v_2}{V} \qquad \dots \qquad \dots \qquad (2)$

Again, if K and k be the bulk moduli of the liquid and of the solid material of the vessel A respectively, we have

$$K=\frac{P}{\frac{v}{V}} \text{ and } k=\frac{P}{\frac{v_2}{V}}$$

or, $\frac{v}{V}=\frac{P}{K}$ and $\frac{v_2}{V}=\frac{P}{k}$

Substituting these values in (2)

$$\frac{v_1}{V}=P\left(\frac{1}{K}-\frac{1}{k}\right)$$

From this K is determined from a previous determination of k.

Regnault's experiment has been more recently extended by Bridgman* up to a pressure of 12000 atmospheres. The liquid under examination was placed in a cylinder made of a special kind of steel. The pressure was exerted by forcing a piston into the cylinder, and measured by observing the change in electrical resistance of a manganin wire placed in the vessel. The change in volume of the liquid was measured from the advance of the piston. Bridgman found the bulk modulus of water was a maximum at 50°C. The value of which is $2{\cdot}3\times10^{10}$ dynes per. sq.cm.

134. Elasticity and Temperature—In general, the moduli of elasticity of a solid decrease with increase of temperature. Thus, for a given stress the strain or deformation increases with the increase of temperature. In other words, the solid passes to more and more plastic state with the rise of temperature.

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* Proc. Amer, Acad, 49, 3, (1918) and 66, 185 (1931).

Andrews working with bent bars at various temperatures has given a relation between Young's modulus and absolute temperature within 150°C of the melting point of the material in an exponential form

$$Y = Y_1 e^{-b_1 T}$$

where b_1 has one value for one-half of the melting point and another for higher temperatures.

An exception is shown by quartz fibre with which the elastic moduli increase with increase of temperature. Thus, with the rise of temperature quartz fibre becomes more elastic.

With liquids, the bulk modulus decreases or the compressibility increases with the rise of temperature. An exception is shown by water with which the bulk modulus is a maximum near about 50°C.

135. Bulk Modulus of Gases—The change of volume for changes of pressure is much larger in the case of gases than in the case of solids and liquids. When a gas is subjected to an increased pressure it undergoes a diminution in volume and simultaneously the gas gets heated in general. If the increase of the pressure takes place slowly, the heat produced gradually passes away. If, on the other hand, the pressure is increased suddenly, the heat produced has hardly any time to pass away. The diminution in volume of a gas for slow compression is larger than for sudden compression; for, the rise of temperature in sudden compression will have its effect and will not allow the gas to undergo as large a diminution in volume as in slow compression. Thus, for the same compressional stress, the volume strain is greater and consequently the bulk modulus is less for slow compression than for sudden compression of the gas. If the heat of compression *fully* passes away and there is no change in temperature of the gas on compression, the state is called **isothermal**. If, on the other hand, the heat of compression *fully* remains lodged in the gas, the state is called **adiabatic.** The bulk modulus of a gas is studied under the two conditions.

(*a*) *Isothermal Elasticity*—The isothermal condition of a *perfect gas* ig given by Boyle's law

$$PV = \text{a constant.}$$

Differentiating with regard to V

$$P + V\frac{dP}{dV} = 0$$

$$\text{or, } P=-V\frac{dP}{dV}=\frac{dP}{-\frac{dV}{V}}$$

dP measures the compressive stress and $-\frac{dV}{V}$ the volume strain, the negative sign indicates that an increase of pressure causes a decrease in volume.

$\therefore \frac{dP}{-\frac{dV}{V}}$ measures the modulus of volume elasticity of the gas.

Thus, the modulus of *isothermal* volume elasticity of a gas is equal to the pressure of the gas.*

(*b*) *Adiabatic Elasticity*—The adiabatic condition of a *perfect gas* is given by the relation,

$$PV^{\gamma}=\text{a constant}$$

Differentiating with regard to V

$$P\gamma V^{\gamma-1}+V^{\gamma}\frac{dP}{dV}=0$$

Dividing by $V^{\gamma-1}$

$$P\gamma+V\frac{dP}{dV}=0$$

$$\text{or, } P\gamma=-V\frac{dP}{dV}=\frac{dP}{-\frac{dV}{V}}$$

which measures the modulus of volume elasticity. Thus, the modulus of *adiabatic* volume elasticity of a gas is equal to γ times the pressure of the gas, where γ=ratio between the specific heats of the gas at constant pressure and at constant volume.†

136. Experiments on Isothermal Elasticity of Gases.

(1) *Boyle*—The behaviour of gases under pressure was first studied by Robert Boyle in 1661. He experimented with air. A glass tube, of the shape of the letter U, one limb of which was

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* For an algebraic treatment of this, see 'A Text-Book of Sound' by the author. Art. 45.

† For an algebraic treatment of this, see author's "A Text-Book of Sound," Art. 45.

shorter than the other, was fixed upright. The shorter limb was closed and a paper scale was pasted on it. The longer limb was open to the atmosphere. Through the open limb mercury was poured into the tube. The volume of the air enclosed at any stage is proportional to the length of the closed limb above the level of mercury in it. The pressure of the air was obtained by adding the difference between the mercury levels in the two limbs to the height of the mercury barometer. Boyle made a series of measurements till the volume of air enclosed was reduced to one-quarter of its original volume and found that the pressure and volume of the gas were in reciprocal proportions. Boyle's law was rediscovered at a later date by the French physicist Mariotte and the law is sometimes spoken of as Mariotte's law. It may be stated as :—

The temperature remaining constant, the volume of a certain mass of gas varies inversely as its pressure.

Since the density of any mass is inversely proportional to its volume, it follows that the density of a gas is directly proportional to its pressure at constant temperature.

(2) *Despretz*—After Boyle, the study of gases was made by Despretz in 1827. He experimented on different gases and compared their behaviour with that of air. Different gases were enclosed in barometer tubes of the same length standing side by side in the same cistern containing mercury, so that the gases occupied the same volume initially. The system was then placed in a strong vessel filled with water and closed by a screw. Pressure was applied to the mercury in the cistern and the mercury was forced up in the tubes. It was found that for the same increase in pressure the volumes of different gases did not decrease by the same amount. It was thus evident that all gases did not obey Boyle's law accurately but whether air obeyed the law or not was left unsettled.

(3) *Regnault*—By his experiments on gases in 1847, Regnault made a great advance in the study of the behaviour of certain gases for pressures between 1 and 30 atmospheres. He took a hard glass tube one end of which was provided with a stop-cock and the other end dipped into a cistern of mercury. The gas under examination was enclosed within the tube provided with a water jacket to keep the temperature of the gas constant. The pressure on the gas was subsequently increased by pumping mercury into the tube until the gas was reduced to half its original volume. To measure the pressure an open tube was fixed in the cistern alongside the experimental tube, in which the mercury rose with increase of pressure. The difference between the levels

of mercury in the two tubes added to the height of the mercury barometer gave the pressure of the gas.

Subsequently fresh gas under this pressure was introduced into the experimental tube through the stop-cock until the gas in the tube occupied the original volume. The stop-cock was closed and the pressure on the gas was again increased until the volume of the gas was reduced to half its volume.

Making a series of experiments between the range of pressures from 1 atmosphere to 30 atmospheres with several gases, Regnault found that for all gases excepting hydrogen, the product of pressure and corresponding volume decreased with the increase of pressure. This showed that these gases were more compressible or less elastic at high pressures than they would be, if they obeyed Boyle's law. With hydrogen, on the other hand, Regnault found that with increase of pressure, the product of pressure and volume increased showing that at high pressure hydrogen was less compressible or more elastic than it would be, if Boyle's law were obeyed.

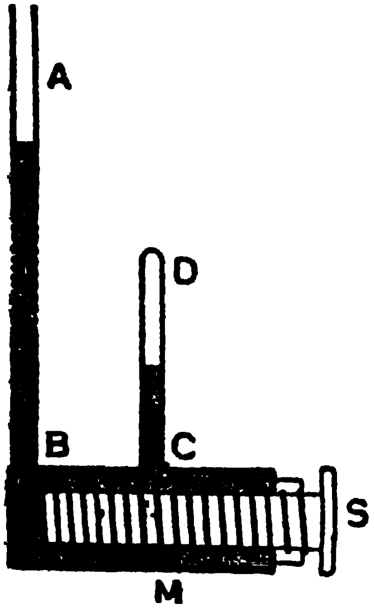

Fig. 80

(4) *Amagat*—Amagat* commenced his important experiments on the elasticity of gases in 1870. He worked at the bottom of a mine. His apparatus, shown in fig. 80, consisted of a pipe *AB*, 300 metres long extending up to the top of the mine. The pipe could be filled to any height with mercury and the pressure at its bottom could be increased thereby. The pressure was measured by observing the compression of nitrogen gas contained in the hard glass tube *CD* which was fixed in the cistern *M* containing mercury, along with the pipe *AB*.

The gas to be experimented upon was enclosed in another tube (not shown in the figure) connected to this cistern and placed in a bath. The bath served to keep the temperature of the gas constant during a series of observations. The temperature of the bath could be changed to carry the experiment at different temperatures of the gas. The volume and temperature of the gas were obtained directly from the experimental tube and the pressure of the gas was determined from the nitrogen manometer.

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* Cailletet also conducted similar experiments about the same time.

The screw S could be employed to force the mercury into the pipe from the cistern M.

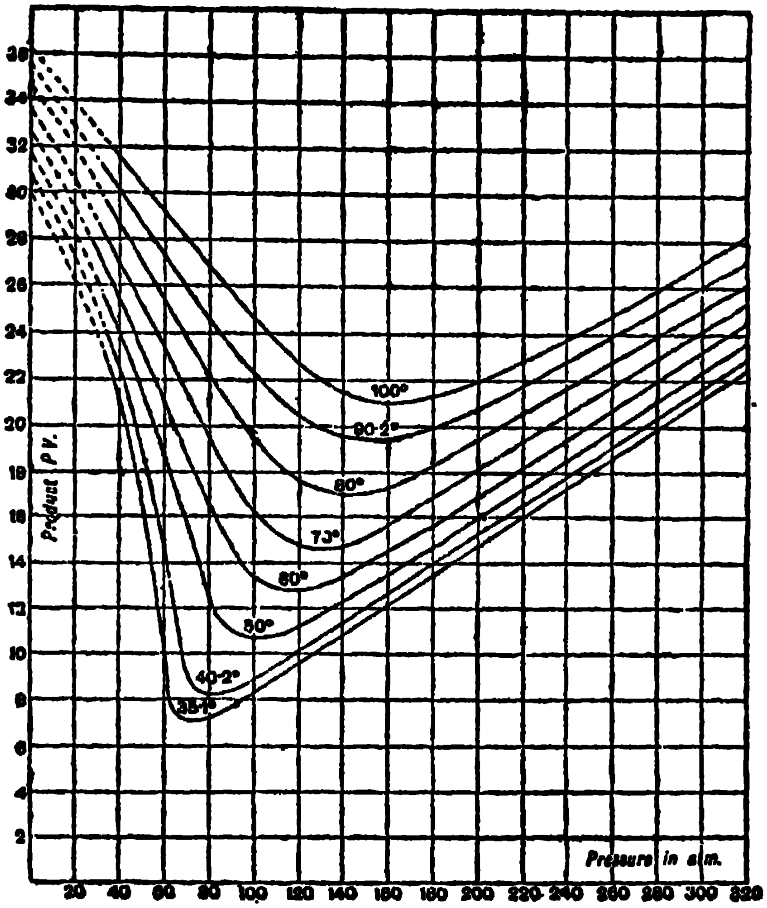

Fig. 81

Amagat experimented with nitrogen, oxygen, air, carbon-monoxide, carbon-dioxide, etc., under pressures exceeding 300 atmospheres. He represented his results on a graph, plotting the product of pressure and corresponding volume against the pressure of the gas. The results of experiments on carbon-dioxide and nitrogen are shown in figs. 81 and 82 respectively.

Amagat found that with all gases excepting hydrogen. the product pv diminishes with increase of p at first until it reaches a minimum, after which the product pv increases indefinitely with increase of pressure. The effect is more marked with ethylene and carbon-dioxide. With hydrogen the product pv increases with p.

Amagat further found that with the former class of gases of which pv falls with increase of pressure, the drop of pv becomes less and less marked, the higher the temperature of the gas as shown in fig. 81. Thus, it can be well surmised that above a critical temperature the curve would show only upward slope like that of hydrogen. With hydrogen it is to be similarly expected that at ordinary temperatures hydrogen is above its critical temperature, and at low temperatures the hydrogen curve would show a sag like that of nitrogen or carbon-dioxide.

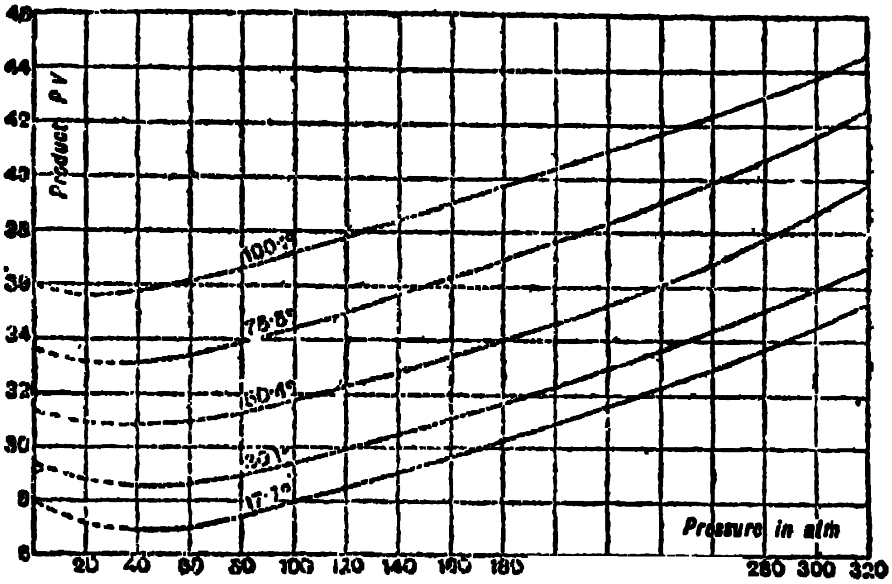

Fig. 82

The experiments of Amagat have been extended to pressures extending up to 3000 atmospheres by Threl-fall and Adair, by a method similar to that used for studying the compressibility of liquids.

Examples

1. Find the force necessary to stretch by 1*mm.* a rod of iron 1 metre long and 2*mm.* in diameter. Also calculate the energy stored in the stretched rod [Young's modulus for iron$=2\times10^{12}$ C. G. S. units] (*Patna 1949*)

Force required$=Y\times$cross-section$\times$strain

$$=2\times10^{12}\times\pi('1)^2\times\frac{'1}{100}\text{ dynes}=\frac{2\times3'14\times10^7}{981\times10^3}\text{ or, 64 kgs.wt.}$$

Energy stored—$\frac{1}{2}\times$stress$\times$strain$\times$volume of the rod (Art. 132, 1)

$$=\frac{1}{2}\times\frac{2\times\pi\times10^7}{\pi('1)^2}\times\frac{'1}{100}\times\pi('1)^2.100=\pi\times10^6\text{ ergs.}$$

2. A uniform rod 120 *cm*, long is clamped horizontally at one end. A weight of 100 *gm*. is attached to the free end. Calculate the depression at a point 90 *cm*. distant from the clamped end. The diameter of the rod is 2 *cm*. Young's modulus of the material of the rod is $1{\cdot}013\times10^{12}$ dynes per sq. cm. and $g=980$ cm/sec². [Bombay 1946]

The equation to the neutral axis for a cantilever is given by

$$Yiy=F\left(\frac{lx^2}{2}-\frac{x^3}{6}\right) \text{ (Art. 129 Eqn. 2)}$$

where y is the depression at a distance x from the clamped end and i, the moment of inertia of the cross-section$=\frac{1}{4}\pi r^4$

$$\therefore\ y=\frac{100\times980\left\{\frac{120\times(90)^2}{2}-\frac{(90)^3}{6}\right\}\times4}{1{\cdot}013\times10^{12}\times3{\cdot}14\times1^4}$$

$$=\frac{100\times980\times(90)^2\times45\times4}{1{\cdot}013\times3{\cdot}14\times10^{12}}$$

$$={\cdot}045 \textit{ cm} \text{ or } {\cdot}45 \textit{ mm}.$$

3. Find the value of Young's modulus from the following experiment. In an experiment the diameter of the rod was $1{\cdot}26$ cm. and the distance between the knife-edges 70 cm. On putting a load of 900 gm. at the middle point the depression was $0{\cdot}025$ cm. Calculate Young's modulus of the substance. (*Agra 1948*)

For a rod supported at the ends and loaded at the middle, the depression δ at the middle is given by

$$\delta=\frac{Fl^3}{12\pi r^4Y} \text{ (Art. 130)}$$

$$\therefore\ Y=\frac{Fl^3}{12\pi r^4\delta}$$

$$=\frac{900\times981\times(70)^3}{12\times3{\cdot}14\times({\cdot}63)^4\times{\cdot}025} \text{ or } 2{\cdot}04\times10^{12} \text{ dynes/sq.cm.}$$

4. A sphere of mass 800 gms. and radius 3 cms. is suspended from a wire of length 100 cms. and radius $0{\cdot}5$ mm. If the period of torsional vibration is $1{\cdot}23$ sec., calculate the rigidity of the material of the wire. (*Bombay 1936*)

The period of torsional oscillation is given by

$$n=\frac{8\pi lI}{T^2r^4} \text{ where } I=M.\ I \text{ inertia of vibrating system, (Art. 127)}$$

$$=\frac{8\times3{\cdot}14\times100\times2\times800\times9}{(1{\cdot}23)^2\times({\cdot}05)^4\times5} \quad \left(\text{as } I=\tfrac{2}{5}\text{ mass}\times(\text{rad.})^2 =\tfrac{2}{5}\times800\times9 \text{ gm. cm}^2\right)$$

or, $7{\cdot}66\times10^{11}$ dynes/cm².

5. Deduce expression for the moment of the couple required to twist the lower end of a rod of circular section by 90°, the upper end being fixed. (*Agra 1945*).

$$\text{Couple required}=\frac{\pi n r^4\theta}{2l}\ \text{(Art 126) (where } \theta \text{ is in radian)}$$

$$=\frac{\pi^2 n r^4}{4l}\ \left(\text{as } \theta=90^\circ=\frac{\pi}{2}\ \text{radian}\right)$$

6. Given that Young's modulus of steel is 2×10^{12} dynes per sq. cm. and the rigidity modulus 8×10^{11} dynes per sq. cm. Calculate Poisson's constant for steel. (*R. U. 1949*)

From relation $Y=2n(1+\sigma)$ (Art 123 *b*)

$$1+\sigma=\frac{Y}{2n}=\frac{2\times10^{12}}{2\times8\times10^{11}}=\frac{5}{4}$$

$$\therefore\quad \sigma=\tfrac{1}{4}=\cdot25.$$

7. A man weighing 70 kilograms stands on the middle of a wooden plank 4 metres long, 30 cms. broad and 4 cms. deep, supported at its ends. What will be his time of swing ? (Young's modulus of wood=10^{11} dynes/sq. cm.

The period of swing is given by

$$T=2\pi\sqrt{KM}$$

where $K=\dfrac{l^3}{4Ybd^3}$ (for a bar supported at ends and loaded at the middle, Art. 131)

$M=70$ kilograms (Mass of the man)

$$T=2\pi\sqrt{\frac{l^3M}{4Ybd^3}}$$

$$=2\pi\sqrt{\frac{(400)^3\times70{,}000}{4\times10^{11}\times30\times4^3}}$$

$$=2\pi\sqrt{\tfrac{7}{1200}}=\cdot48\ \text{sec.}$$

Exercise

1. Explain the terms,—'Stress,' 'Strain,' 'Young's modulus' and 'Pisson's ratio.'

A steel wire of length 2 metres and of diameter 0·52 mm. is suspended from an unyielding support and loaded with 5 kilogrammes. An elongation of 2·81 mm. is observed. Calculate the coefficient of longitudinal elasticity of steel. (*C. U. 1935*)

2. What do you understand by the elastic constants of a solid? What is Poisson's ratio?

Describe an experiment to measure the simple rigidity of the material of a wire deducing the formula. (*C. U. 1937*)

3. Define the coefficient or rigidity. Describe a method for finding the coefficient of rigidity of a wire and deduce the formula to be used. (*C. U. 1939, 45*)

4. Give an account of Regnault's and Amagat's experiments on Boyle's law at high pressures. Mention the conclusions drawn from the experiments. (*C. U. 1939*)

5. Define and explain Young's modulus, modulus of rigidity and Poisson's ratio and derive an expression showing the relation between them. Hence, show that Poisson's ratio always lies between $\frac{1}{2}$ and -1. (*C. U. 1942*)

6. Explain what is meant by the modulus of rigidity and find its dimensions.

Describe a method of finding experimentally the modulus of rigidity of a solid and give the theory of the method. (*C. U. 1943*)

7. What do you understand by stress, strain and elastic limit? Explain what is meant by torsional rigidity and describe a method for finding the same, deriving the formula you employ. (*C. U. 1945*)

8. Explain the terms :—Stress. strain, Young's modulus, Poisson's ratio, bulk and rigidity moduli. Show that the value of Poisson's ratio must lie between (-1) and $(+\frac{1}{2})$. (*C. U. 1947*)

9. Define an isotropic body. Explain what is meant by the simple rigidity of a solid? How would you measure it experimentally? (*C. U. 1949*)

10. What do you understand by the terms *stress*, *strain* and *co-efficient of elasticity*?

You are given 200 c.c. of air at a pressure due to 760 mm. of mercury. On increasing the pressure by that due to 1 mm. of mercury, without change of temperature, the volume is observed to decrease by 0·268 c.c. Find the coefficient of volume elasticity of the gas. (*C. U. 1952*)

11. Define stress, strain and Young's modulus. What is Poisson's ratio? A copper wire 2 metres long and 0·5 mm. in diameter supports a mass of 10 k.g. It is stretched by 2·38 mm. Calculate the Young's modulus of the wire. (*C. U. 1953*)

12. Explain what is modulus of rigidity? One end of a straight wire of length l and of external and internal radii R and r respectively is fixed. Find the couple required to twist the other end through an angle θ. What is the work done in producing the twist? (*N. U. 1950*)

13. Describe an experiment for determining the modulus of rigidity of a wire. Derive the formula used. (*L. U. 1947, P. U. & E. P. U. 1951*)

14. Define modulus of rigidity of a solid substance. Calculate the couple necessary to twist a cylindrical rod through an angle at one end, the other end being clamped. (*R. U. 1952*)

15. Describe how you would measure the modulus of rigidity of the material of a wire. Give the theory of the method. (*L. U. 1949*)

16. Define (*i*) Young's modulus Y, (*ii*) modulus of rigidity n (*iii*) bulk modulus K, and (*iv*) Poisson's ratio σ. Show that for a homogeneous isotropic substance $Y=2n(1+\sigma)$. (*P. U. & E P. U. 1950*)

17. A thin uniform beam of rectangular section is clamped horizontally at one end and loaded at the other. Find the relation between the bending moment and the radius of curvature at any point of the bent beam. Determine also the depression of the loaded end. (*N. U. 1955*)

18. A uniform rod is suspended by a thin wire passing through the centre of gravity of the rod. If the rod executes small angular oscillations in a horizontal plane, find an expression for the period of these oscillations. Show how you would take observations to get an accurate value of the rigidity modulus of the material of the wire. (*N. U. 1954*)

19. Express the modulus of rigidity in terms of Poisson constant. Calculate the Poisson constant for steel, given that its Young's modulus is 2×10^{12} dynes/sq. cm. and the rigidity modulus is 8×10^{11} dynes/sq. cm. (*R. U.* 1949)

CHAPTER X

SURFACE TENSION

137. Surface Forces—Many familiar observations show that a liquid surface exerts forces on its boundary. These forces are *tangential* to the liquid surface, *normal* to the liquid boundary and are in such a *direction* that the surface area of the liquid tends to contract. The forces, thus, endow the liquid surface with the properties of a stretched elastic membrane. The forces are present on the surface alone and not within the interior of the liquid mass. Some observations can be made in the following way.

Make a circular loop of wire and tie a thin cotton or silk thread loosely across it. Dip it into a solution of soap and water and take it out. A thin film of the solution will spread over the loop and the thread will lie across the film in an irregular manner. If one portion of the film is now broken by touching it, it will be found that the thread will be at once pulled towards the other portion and will be placed in the form of an arc of a circle, as if the film was tightly stretched and was trying to contract so as to occupy a smaller area. It should be observed that the force acting upon the thread is entirely distinct from the force of gravity on it; for, if the loop dipped in the soap solution is held

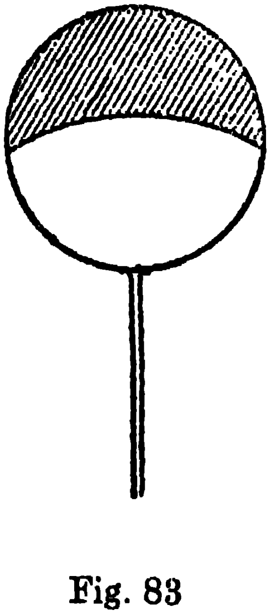

Fig. 83

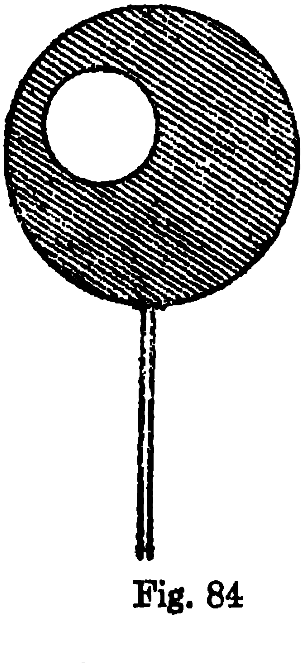

Fig. 84

vertically and the lower portion the film of it is broken, the thread will be pulled in the upward direction (fig. 83); although the force of gravity pulls the thread in the downward direction.

Instead of tying the thread across the loop, first dip the loop into the solution to obtain a film as before. On this film place a closed loop of cotton or silk and finally break the film inside the loop of cotton or silk. It will be found that the loop of cotton or silk is at once drawn into a circle (fig. 84). The effect is similar to the previous one, showing that the film tends to shorten its area.

Of all plane figures having a given perimeter, the circle has the maximum area. Thus, the loop of cotton is drawn by the film so that the area within it is a maximum and hence the area of the film is a minimum. In other words, the film of liquid tends to occupy the least area possible.

188. Definition of Surface Tension—The surface of a liquid mass behaves somewhat like a stretched elastic membrane. If on the surface of a liquid a straight line is imagined to be drawn, the portions of the liquid surface on both sides of the straight line tend to draw away from each other. The consequent split of the liquid surface is prevented by the force of attraction lying between molecules on opposite side of the straight line. The force, with which each portion of the surfaces on opposite sides of the straight line of *unit length* is pulled away, is called the *surface tension* of the liquid.

It must be observed that this force acts *perpendicular* to the straight line and *tangential* to the liquid surface.

The surface tension of the liquid depends, to some extent on the medium in contact with the surface of the liquid. The surface tension of water in contact with air is nearly 74* dynes per centimetre while that of water in contact with its vapour is nearly 71·4* dynes per centimetre.

The surface tension also depends upon the temperature of the liquid. It follows from *Eotvos'* rule that the surface tension of a liquid at a temperature t is directly proportional to $(t_c - t)$ where t_c is the critical temperature of the liquid.

189. Potential Energy of a Liquid due to Surface Tension—A liquid surface possesses potential energy just as a stretched elastic membrane possesses potential energy. To calculate the potential energy of liquid surface, consider a rectangular framework $ABCD$ (fig. 85) of which the arms AB, AD, and BC are fixed and CD can slide along the arms AD and BC. If a film of liquid, say a film of soap solution, is contained in the rectangle $ABCD$, the film will exert forces on the sides of the framework; the direction of the force on any of the sides will be *perpendicular* to the side and *tangential* to the surface of the film *towards* the interior of the film. These forces will have the same magnitude and will act similarly, whatever be the position of the movable arm CD. Hence, the potential energy, per unit area of the film, will remain unaltered, if the film is stretched so as to occupy a greater area. The work done in stretching will supply potential energy for the extended area. Thus, the work done by an external agency in stretching the film through unit area, against the forces due to surface tension, is a measure of the potential energy per unit area of the film.

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*From Kaye and Laby's table of Physical Constants.

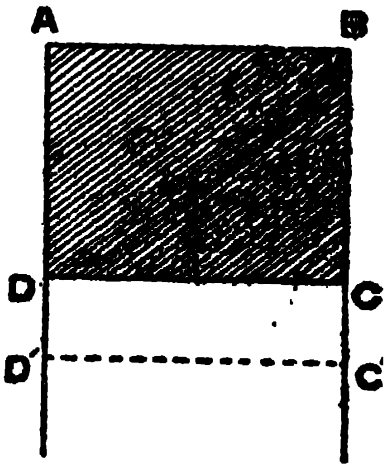

Fig. 85

The liquid film has evidently two surfaces and if T is the surface tension of the film, the force on unit length of the arm CD due to each surface is T. Hence, the total force on CD directed perpendicularly inwards$=2T.CD$. If the film is stretched by an external force to the position $C'D'$, the work done by the external agency against the force due to surface tension$=2T.CD.CC'$. Again, the increase in area of both sides of the film due to the stretching$=2CD.CC'$. Thus, the work done in stretching the film through unit area, which is a measure of the potential energy per unit area of the film

$$=\frac{2T.\ CD.\ CC'}{2.\ CD.\ CC'}=T.$$

*Therefore the potential energy per unit area of the surface is equal to the surface tension of the liquid.**

140. Problems treated by Surface Tension—

Many important problems can easily be treated by the conception of surface tension as potential energy per unit area of the surface.

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*Thermo-dynamical considerations, however, show that, generally, the potential energy per unit area is greater than the surface tension. (See Newman and Searle's General Properties of Matter.)

(1) If a mixture is made of suitable quantities of water and alcohol, so as to have the same density as that of olive oil and if a small quantity of olive oil is dropped into the mixture, it will remain somewhere within the mixture in suspension. The effect of gravity on the olive oil is counter-balanced by the buoyancy of the mixture. To determine the shape which the oil will take up, we shall be guided by the universal principle that—*a mechanical system is in stable equilibrium when its potential energy is a minimum.* Thus, a compressed clock-spring releases itself and keeps the clock going.

Since there is no effect of gravity on the oil, it possesses potential energy due to surface tension only. Thus, the oil will take up a shape in which its potential energy due to surface tension is a minimum, *i.e.*, in which it has the least area of its surface. Now a sphere is the figure, which for a given volume has the smallest surface area. Hence, the shape of the oil within the mixture is spherical.

(2) If a quantity of liquid is dropped on a solid surface which the liquid does not wet, to determine the shape of the liquid we shall be guided by the same universal principle that the potential energy of the liquid must be a minimum in the shape it takes up. If the quantity of liquid is large and if it takes up a spherical shape instead of spreading on the solid surface, its centre of gravity rises well above the solid surface. Thus, although in the spherical shape it possesses the least potential energy due to surface tension, yet owing to a large gain of potential energy due to gravity, it cannot remain in the spherical form. Hence, the liquid spreads on the solid. Thus, a large quantity of mercury spreads on a glass plate instead of collecting in a drop.

If, on the other hand, the quantity of liquid dropped on the solid surface is small, its potential energy due to gravity is insignificant compared to that due to surface tension.

The liquid will take up a shape in which its potential energy due to surface tension is a minimum. Thus, instead of spreading on the solid surface, it will collect in nearly spherical droplets. Dewdrops are very conspicuous examples of this. The rainbow affords a proof that the raindrops are spherical. The formation of the rainbow and its colouring are based upon the spherical forms of raindrops. Again, in the manufacture of shots, molten lead is poured down in a fine stream from the top of a high tower. The stream breaks up into small drops which cool and solidify while falling.

(3) This method of energy consideration can be applied to determine whether or not a liquid will spread when placed on the surface of another. Let a quantity of a liquid *A* be placed on the surface of another liquid *B* of higher specific gravity with which it does not mix. If the spreading of the liquid *A* over *B* entails an increase in potential energy of the system, the liquid *A* will collect in a spherical drop on the surface of *B*. On the other hand, if the spreading entails a decrease in the potential energy of the system, the liquid *A* will spread uniformly over the entire surface of *B*.

Here, there are three surfaces of contact : that of the liquid *A* and air, that of the liquid *B* and air and that between the two liquids. Suppose the area of contact between the liquid *A* and the liquid *B* is increased by an amount *S*. In doing so the area of contact between *A* and air is increased by *S*, that between the two liquids is increased by the same amount, whereas that between *B* and air is decreased by *S*. If T_A represents the surface tension between the liquid *A* and air ; T_B the surface tension between the liquid *B* and air and T_{AB} the surface tension between the two liquids, the gain of potential energy of the system in the above process is

$$(T_A+T_{AB}-T_B)S$$

If $T_B>T_A+T_{AB}$, the above expression is negative. Hence, the spreading involves a decrease in potential energy of the system. Thus, the liquid *A* will spread on the surface of the liquid *B*.

If, on the other hand, $T_A+T_{AB}>T_B$, the expression is positive. Hence, the spreading is associated with an increase of potential energy of the system. Therefore, the liquid *A* will collect in a drop on the surface of *B* instead of spreading on it.

If T_A+T_{AB} is greater than T_B, evidently a triangle can be constructed with its sides proportional to T_A, T_{AB} and T_B. Such a triangle is called **Neumann's triangle**. Experimental values of T_A, T_{AB} and T_B show that no such triangle can be drawn for any two liquids. Hence, one liquid cannot collect in a drop when placed on another but spreads on it. A familiar example of this is the spreading of oil over a sheet of water. An apparent exception is shown by drops of water on the surface of mercury. This is due to the fact that the surface of mercury is contaminated with grease. Quincke has shown that pure water spreads over the surface of pure mercury.

141. Angle of Contact—If a glass plate is plunged vertically in a liquid, such as ether, water, etc., it is found that the liquid, where it touches the surface of glass, is drawn up (fig, 86). On the other hand, opposite effect is shown, when the glass plate is similarly plunged in mercury which is depressed where it touches the surface of glass (fig. 87). Consider a point P on the line at which the liquid meets the solid surface. The angle APB which PA, the tangent to the liquid surface at P, makes with the immersed solid surface is called the *angle of contact* between the liquid and the solid. A section of the liquid surface by a vertical plane perpendicular to the glass-plate gives the form of the liquid meniscus at the place of contact and is known as the *capillary curve.*

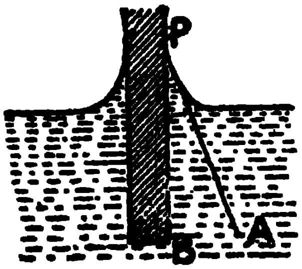

Fig. 86

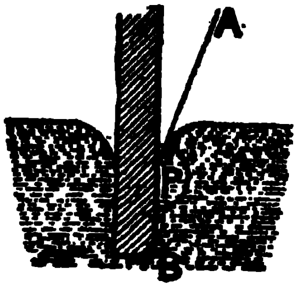

Fig. 87

The cause of the change of curvature of a liquid near place of contact with a solid is due to the fact that a molecule of liquid, near the place of contact between the liquid and the solid. is under a condition different from that of a molecule of the liquid remote from the solid body.

The forces, acting on a molecule at P (figs. 88 and 89) of the liquid near the place of contact with solid surface, are twofold :—

(1) A force of adhesion exerted by the solid. This force is directed along PQ perpendicular to the solid and its magnitude depends upon the nature of the solid and the liquid. It is very intense between glass and water but not so between glass and

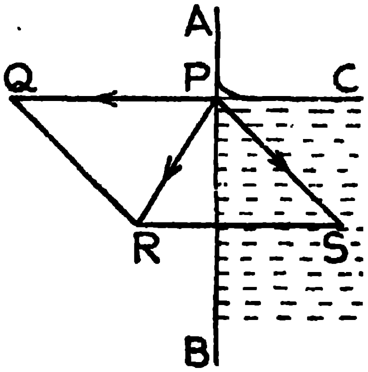

Fig. 88

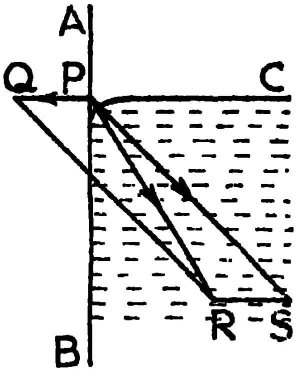

Fig. 89

mercury. This is evident from the fact that an enormous normal force is required to separate two glass plates intervened by a film of water. (2) The other force on the liquid molecule at P is the resultant of forces of cohesion exerted by the rest of the molecules of the liquid. This force arises from the molecules lying within a short range of P called the range of molecular attraction. It is almost absent when the distance of the attracting molecule exceeds this range (Art. 150). It does not obey the *distance law* of gravitation. This force is intense between the molecules of a heavy liquid, such as mercury, but not so between the molecules of water. The resultant of the cohesive forces due to various molecules on the molecule at P is shown by PS. This is evidently inclined at 45° to the glass plate.

Fig. 88 shows the directions and rough relative magnitudes of the adhesive PQ and resultant PS of cohesive forces on a water molecule at P and fig. 89 is a similar diagram for a mercury molecule at P. The resultant of the two is shown by PR.[*] The free surface of a liquid at any point is perpendicular to the resultant force at that point as otherwise there will be a tangential component of the resultant force which a liquid molecule is unable to bear. Hence, the surface of water is concave upwards where it is in contact with glass (fig. 88) and the surface of mercury is convex upwards at its place of contact with glass (fig. 89).

At a point remote from the solid, the adhesive force on a liquid molecule is absent, and the liquid molecule is under the action of the cohesive force due to the rest of the liquid, acting vertically downwards and the free surface of the liquid is, in consequence, horizontal.

142. Capillarity—If a capillary glass tube, open at both ends, is dipped vertically into ether or water, the liquid is found to rise within the tube. The smaller the cross-section of the tube the higher is the level to which the liquid rises in it. The rise of the liquid in the tube is due to the surface tension of the liquid. Ether or water has an acute angle of contact with glass. Thus, within the tube, the surface of the liquid has its concavity turned upwards. This causes an increase in surface area of the liquid within the tube and a consequent increase in potential energy of the surface (Art. 139). The bottom of the meniscus rises in order to minimise the surface area and again the periphery of the liquid surface rises more to maintain the same angle of contact, with a consequent rise of the bottom of the meniscus. This goes on until the force due to surface tension is balanced by the weight of the liquid that rises within the tube.*

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[*] For an elaborate account of this, the reader is referred to Newman and Searle's Properties of Matter, Art. 81.

When the tube is similarly dipped into mercury, instead of an elevation of mercury within the tube, a depression is found to take place. Mercury has an obtuse angle of contact with glass and within the tube the surface of mercury has its concavity turned downwards. This increases the surface area of mercury within the tube with a consequent increase in potential energy of the surface. The top of the meniscus comes down to minimise the surface-area until the force due to surface tension is balanced by the force due to the excess of hydrostatic pressure at the lower level. This phenomenon of elevation or depression of a liquid, when a capillary tube is dipped vertically into it, is known as *capillarity.*

143. Pressure within a Spherical Soap Bubble—When a spherical soap bubble is blown†, the pressure inside the bubble is in excess of the pressure outside it. Due to the excess of internal pressure, the bubble has a tendency to expand. The surface tension of soap causes the bubble to contract. Equilibrium is obtained when the force due to the excess of internal pressure on the bubble is balanced by the force due to surface tension on it.

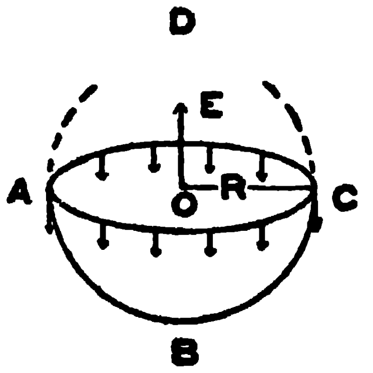

Fig. 90

To obtain a relation between the excess of internal pressure and the surface tension, consider a spherical soap bubble $ABCD$ (fig. 90). Divide the soap bubble into two halves by an imaginary plane AC passing through its centre O. The force exerted by one half, say ABC of the bubble on the other half ADC are as follows :—

(1) Due to the excess of internal pressure, the half ABC exerts a force on the other half. The direction of the force due to this excess of internal pressure is along OE normal to the plane AC through its centre O. If p is the excess of internal pressure, the force through each unit area of the plane AC is p. If the bubble has a radius R, the area of the plane surface is πR^2 and the force due to the excess of internal pressure is $\pi R^2 p$.

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*For an explanation of this from surface tension See Art. 146, I.

†A spherical bubble may be blown by dipping a glass tube into a little soap solution and then by blowing through the tube. For blowing bubbles of various shapes the student may consult Professor Boys' volume on soap bubbles.

(2) Due to surface tension, the portion ABC exerts a force on the boundary of the plane surface AC. If T represents the surface tension, the force on each unit length of the boundary is $2T$, as the film has two surfaces. The directions of these forces are *normal* to the bounding line and *tangential* to the surface as shown by the small arrows. Since the length of the boundary is $2\pi R$, all these forces combine into a resultant force of magnitude $2T.2\pi R$ or, $4\pi RT$ opposite to OE. Hence, for equilibrium of the bubble,

$$\pi R^2 p=4\pi RT \quad \text{or,} \quad p=\frac{4T}{R}$$

The excess of internal pressure is inversely proportional to the radius of the bubble. Hence, for the same material of the bubble the product $p.R$ is a constant. Or, the relation between p and R is given by a rectangular hyperbola.

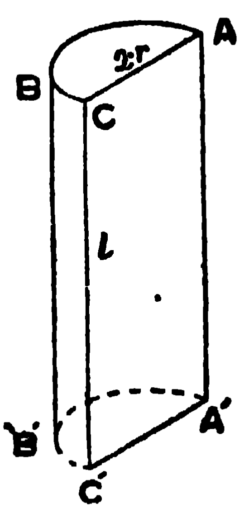

Fig. 91

144. Pressure within a Cylindrical Soap Bubble.

The pressure within a cylindrical soap bubble* is in excess of the external pressure on it. To obtain the magnitude of the excess of internal pressure in terms of surface tension, consider a length l (fig. 91) of the film between two planes ABC and $A'B'C'$ perpendicular to the axis. Divide the portion thus obtained into two halves by a plane $ACC'A'$ through the axis If the excess of pressure inside the bubble is p, the force due to the half $ACBB'C'A'$ on the other half acting through the middle point of the plane $ACC'A'$ normally outwards$=p\times$area of the plane$=p\times 2rl$, where r is the radius of the cylindrical bubble. If T is the surface tension of the film, the force on the bounding lines AA' and CC' of the plane (remembering that the film has two surfaces) combine into a resultant of magnitude $2T.\ (l+l)$ or $4Tl$. This force acts through middle point of the plane surface normally inwards Hence, for equilibrium of the bubble,

$$p\times 2rl=4Tl$$

$$\therefore\ p=\frac{2T}{\ }$$

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*A cylindrical soap bubble is obtained by dipping two rings of wire together in a soap solution. By pulling the rings apart, the film extending between them will be cylindrical.

145. Pressure within a Curved Liquid Membrane—The difference of pressure between the two curved surfaces of the general type can be deduced from following considerations Let *ABCDA*(fig. 92) be a small portion of a rectangular curved surface with sides of length x and y. Let the membrane be so curved that the radii of longitudinal and transverse bending are turned in the same direction, it is called a *synclastic* surface. Let the radii at A and B of the bending along x-direction meet at O and be of length r_1. Similarly, the radii at B and C of bending along y-direction meet at Q and be of length r_2. Then r_1 and r_2 are called the *principal radii of curvature.* The equilibrium of the membrane is established in the following way :—There will be an excess of pressure within the concave side of the membrane and the force of expansion due to this excess of pressure will be balanced by the force of compression on the membrane due to surface tension.

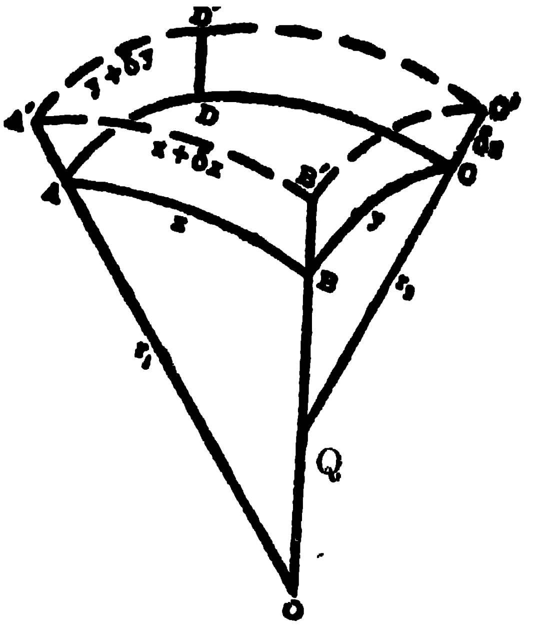

Fig. 92

To co-relate these two effects let the surface area xy be given a normal displacement δz so as to take up the position $A'B'C'D'A'$. The sides x and y become $x+\delta x$ and $y+\delta y$ respectively. The work done by the excess of inside pressure $p = p \times$ area of the membrane $\times$ normal displacement $= p.\ xy.\ \delta z$ ... ... (1)

The increase in surface area of the film

$$=(x+\delta x)(y+\delta y)-xy=x\delta y+y\delta x$$

(neglecting the product $\delta x.\delta y$)

∴ the increase in free surface energy of the film

$$=2T(x\delta y+y\delta x) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where T is the surface tension of the film which has two surfaces.

Equating (1) and (2)

$$pxy.\delta z=2T(x.\delta y+y.\delta x) \quad \ldots \quad \ldots \quad (3)$$

Again, from similar triangles ABO and $A'B'O$

$$\frac{x+\delta x}{x}=\frac{r_1+\delta z}{r_1} \text{ or, } \frac{\delta x}{x}=\frac{\delta z}{r_1} \text{ or, } \delta x=\frac{x}{r_1}\delta z$$

Similarly, from similar triangles BQC and $B'QC'$

$$\delta y = \frac{y}{r_2}\delta z$$

Substituting the values of δx and δy in eqn. (3)

$$pxy\delta z = 2Txy\ \delta z\left(\frac{1}{r_1}+\frac{1}{r_2}\right) \text{ or, } p = 2T\left(\frac{1}{r_1}+\frac{1}{r_2}\right) \quad \ldots \quad (4)$$

If there is only one surface, such as in the case of a liquid drop or an air bubble within a liquid,

$$p = T\left(\frac{1}{r_1}+\frac{1}{r_2}\right) \quad \ldots \quad \ldots \quad (5)$$

Special cases :—(1) a spherical soap bubble

$r_1 = r_2 = r$ (the radius of the bubble)

$$\therefore \quad p = 2T\left(\frac{1}{r}+\frac{1}{r}\right) = \frac{4T}{r} \quad \ldots \quad \text{(from eqn. 4)}$$

(2) A drop of a liquid or an air bubble within a liquid

$r_1 = r_2 = r$ (radius of the drop or the bubble)

$$p = T\left(\frac{1}{r}+\frac{1}{r}\right) = \frac{2T}{r} \quad \ldots \quad \text{(from eqn. 5)}$$

(3) A cylindrical soap bubble

$r_1 = r$ (radius of the cylindrical bubble),

$r_2 = \infty$ (as the end surfaces are plane)

$$\therefore \quad p = 2T\left(\frac{1}{r}+\frac{1}{\infty}\right) = \frac{2T}{r}. \quad \ldots \quad \text{(from eqn. 4)}$$

If the concavities of longitudinal and transverse bending of the membrane are turned in opposite directions, the surface is called an *anticlastic* surface. So that if r_1 is positive, r_2 is negative and the excess of inside pressure is given by

$$p = 2T\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

146. Measurement of Surface Tension—

(1) Method of Capillary Elevation—A thermometer tubing 15 cm. long is taken and its bore carefully washed first with sulphuric acid and then with a strong solution of caustic soda and finally by passing a stream of tap water through the bore for several minutes. Tap water is preferable to distilled water as the latter is very apt to be contaminated with grease, which would be fatal to the accuracy of the experiment.

The liquid is taken in a beaker previously cleansed in the same way as before and the tube is dipped vertically in it (fig. 93). To ensure proper wetting of the tube by the liquid, the tube should at first be dipped deeper into the liquid than it will be when fixed. The elevation of the liquid is measured with the help of a microscope. As it is difficult to focus the microscope on the level of liquid in the beaker, a needle is fixed alongside the tube with the help of an india-rubber ring and the two together arranged vertically with the help of a plumb line. The tube is dipped into the liquid so that the lower end of the needle just touches the surface of the liquid in the beaker. The tube is held in position by a wooden stand. The arrangement is shown in fig. 93. The microscope is first focussed on the lower surface of the meniscus of the liquid in the tube and a reading corresponding to this, is taken on the vertical scale attached to the microscope. The microscope is next focussed on the top of the needle and its reading noted as before. The difference between these two readings added to the length of the needle, if the top of the needle is below the meniscus as shown in the figure, gives the height of the lower meniscus of the liquid in the tube above the free surface of the liquid in the beaker.*

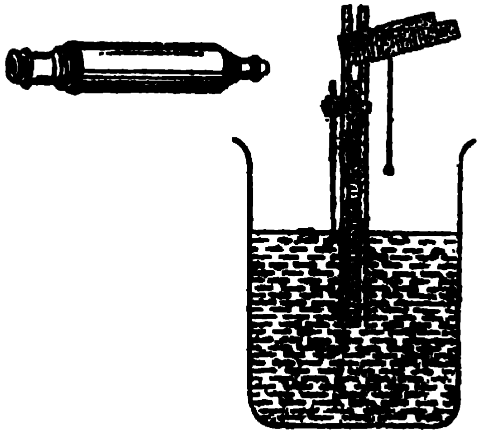

Fig. 93

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* If the top of the needle is above the meniscus of the liquid in the tube, the difference between the microscope readings is evidently to be subtracted from the length of the needle.

If r represents the radius of the capillary tube near the meniscus, the liquid meets the tube along the circumference of a circle known as the *circle of contact.* The force on a short element of this circumference at P (fig. 94) due to surface tension acts tangential to the liquid meniscus *towards* the interior of the liquid ; the force is represented by PQ. This force is exerted by the liquid on the wall of the tube. By the third law of motion, the tube exerts a force at P on the liquid equal and opposite to PQ represented by PQ'. This latter force acting on the liquid, keeps the liquid column in the tube against the force of gravity acting on the liquid.

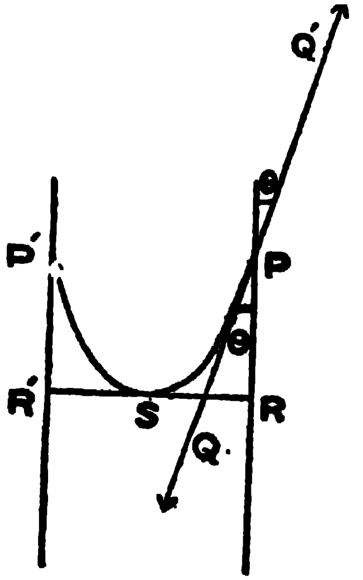

Fig. 94

If the angle of contact between the liquid and glass be θ, the vertical component of the force on each unit length of the bounding line of the liquid meniscus is $T\cos\theta$ and the total vertical upward force on the boundary line of the liquid meniscus

$$=2\pi r.\ T\cos\theta \quad \ldots \quad \ldots \quad (1)$$

The weight of the cylindrical column of liquid of radius r and height h

$$=\pi r^2 h\rho g \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

where ρ is the density of the liquid and g, the acceleration due to gravity.

Since the two forces (1) and (2) balance,

$$2\pi r\ T\cos\theta=\pi r^2 h\rho g$$

$$\text{or,}\quad T=\frac{rh\rho g}{2\cos\theta} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

A simplification of the above relation can be made in the case of liquids, such as water, alcohol, chloroform etc., for which the angle of contact θ is nearly zero. In these cases, $\cos\theta=1$. Hence, from eqn. (3),

$$T=\frac{rh\rho g}{2} \quad \ldots \quad \ldots \quad (4)$$

This ignores the weight of the liquid contained in the tube above the lower meniscus RSR' of the liquid. For the above liquids, since the contact-angle is zero, the meniscus is hemispherical of radius r and the volume of the liquid above the lower meniscus is evidently the difference between the volumes of the cylinder $PP'R'R$ and the hemisphere PSP' which

$$=\pi r^2.r-\tfrac{2}{3}\pi r^3=\tfrac{1}{3}\pi r^3$$

If this liquid is distributed uniformly within the tube, the height to which it will rise will be given by h' where

$$\pi r^2 h'=\tfrac{1}{3}\pi r^3$$

whence $$h'=\tfrac{1}{3}r$$

Thus, the upward force arising out of surface tension balances the weight of a column of liquid of height $h+h'$ or $h+\frac{1}{3}r$. Making this correction, the relation (4) becomes

$$T=\frac{r\rho g}{2}\left(h+\frac{1}{3}r\right)$$

As the height to which the liquid rises depends upon the radius at the place of contact, the radius r should be measured at the place of contact. For this, the tube should be broken near the meniscus and its radius at this point measured with a travelling microscope. As tubes are more of elliptic section, it is better to measure the radius in two rectangular directions and the mean value taken for r.

When using water, tap water should be taken in preference to distilled water for reasons already mentioned.

Equation (4) shows that for the same liquid the product $r.h$ is a constant. This is known as *Jurin's* law.

(2) Measurement of the Pressure-excess inside a Spherical Bubble—

The result obtained in Art. 143 is directly applicable to the measurement of the surface tension of soap solution. A spherical soap bubble is blown at the end of a glass tube having the shape of the letter T. One arm of the tube is connected to a special type of manometer suitable for measurement of small pressure changes. The manometer contains a liquid of low specific gravity, such as xylene, and its two limbs instead of being parallel are inclined at an angle. A small change of pressure at one limb will cause a large shift of the liquid which adds to the sensitiveness of the apparatus. The bubble is blown at one arm of the T-tube by dipping into soap solution and blowing gently through the third arm which can be closed by a stop cock. The excess of internal pressure is measured by the manometer. The diameter of the bubble is measured by a travelling microscope both in horizontal and vertical directions and the mean value of the radius R is determined. The surface tension of soap solution is calculated from the relation

$$p=\frac{4T}{R} \qquad \text{Art. 143}$$

(3) **Jaegar's Method**—Surface tension measurement, such as by method of capillary elevation, involves an uncertainty regarding the value of the angle of contact. Jaegar's method is independent of the angle of contact. Water or air is entered into a large vessel A (Fig. 95) from which air is let out by opening the tap T fairly at a uniform rate, to the open manometer M and the tube BC having a narrow orifice at the end C. The tube BC is dipped vertically in a vessel containing the liquid under examination. Opening the tap a bubble is formed at C the pressure of which increases until it breaks with a corresponding change in the manometer reading. Bubbles are formed at the rate of one per sec. and let the maximum pressure-difference as indicated in the manometer is h. The maximum pressure p_1 inside the bubble is thus given by

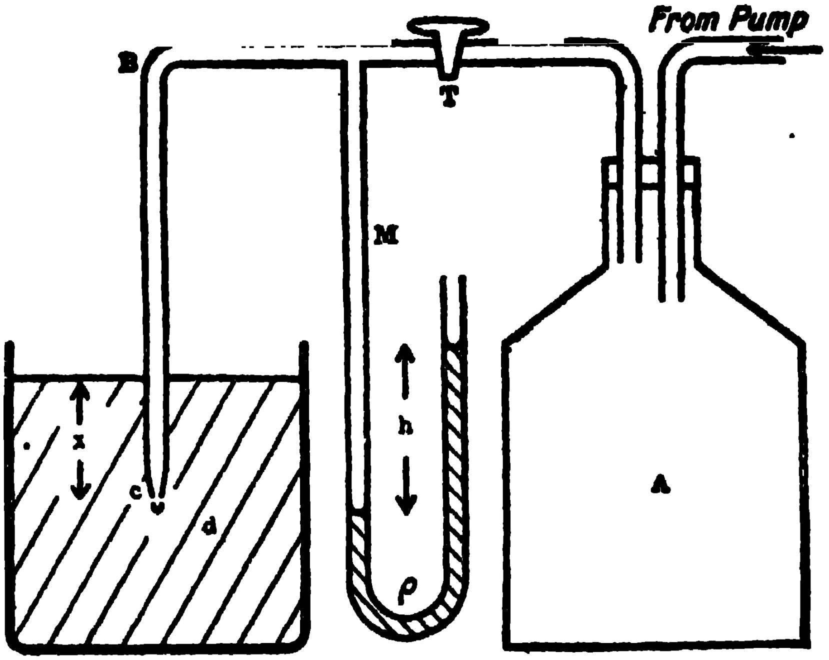

Fig. 95

$$p_1=B+h\rho g \quad \dots \quad \dots \quad (1)$$

where B=barometric pressure and ρ, the density of the liquid used in the manometer.)

Also if the end C of the tube is dipped to a depth x in the vessel, the pressure p_2 outside the bubble is the hydrostatic pressure at the depth x of the liquid and is given by

$$p_2=B+xdg \qquad \dots \qquad \dots \qquad (2)$$

where d=density of the liquid in the vessel

From (1) & (2) $p_1-p_2=(h\rho-xd)g$

Again, this excess of the pressure within the drop above that outside is given by

$p_1-p_2=\frac{2T}{r}\left(\text{Art. } 145{,}2\right)$ where T is the surface tension of the liquid in the vessel and r=radius of the bubble,

$$\therefore \frac{2T}{r}=(h\rho-xd)g \text{ or, } T=\frac{gr}{2}(h\rho-xd)$$

The radius of the bubble r is taken as the radius of the orifice of the tube. Inaccuracy of the result arises as the instability of the bubble is not reached when its radius is equal that of the orifice.

147. Surface Tension and Angle of Contact of Mercury—For substances, such as mercury, which do not wet glass, the surface tension and angle of contact can be found from observations made on a large drop of it. Let OAB (Fig. 96) represent the section of one-half a large mercury drop on a glass plate BB' by a vertical plane OAB' through the middle of it. The drop is so large that the upper surface of it may be regarded sensibly plane. Let a horizontal line through O represent the X axis and the vertical line OB' the Y-axis. Draw a tangent to any point P on the periphery making an angle φ with the X-axis. Let the co-ordinates of P be (x, y), then PP' drawn perp. to OB' is x and OP' is y. The pressure p at a point close to P just within the drop is given by $p=p_0+y\rho g$...(1)

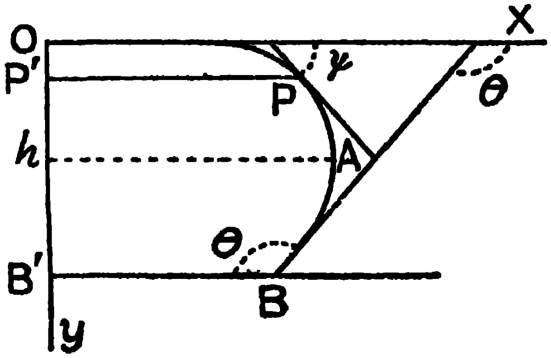

Fig. 96

Where p_0=atmospheric pressure and ρ, the density of mercury. Again, since the surface of the drop is synclastic the excess of pressure just within the drop at P over that outside

$$p-p_0=T\left(\frac{1}{r}+\frac{1}{R}\right) \qquad \dots \qquad \dots \qquad (2) \quad \text{Art. 145 eqn. 2}$$

Where T=surface tension of mercury and r and R are the radii of curvature in and perp. to the plane of the diagram. The drop being large R is large compared to r and eqn. (2) becomes

$$p=p_0+\frac{T}{r} \qquad \ldots \quad (3)$$

From (1) and (3), $y\rho g=\frac{T}{r}$ ... (4)

Again, the curvature $\frac{1}{r}=\frac{d\varphi}{ds}$ where ds is an element of the periphery of the drop at P in the plane of the diagram and φ is the angle which the tangent at P makes with the positive direction of X.

Also $\frac{d\varphi}{ds}=\frac{d\varphi}{dy}\cdot\frac{dy}{ds}=\frac{d\varphi}{dy}\sin\varphi$

$\therefore$ from (4), $y\rho g=T\frac{d\varphi}{dy}\sin\varphi$ or, $\rho gy dy=T\sin\varphi d\varphi$...(5)

Integrating,

$$\int_0^h \rho gy dy=\int_0^{\pi/2} T\sin\varphi d\varphi$$ where h represents the depth of a point A of the drop where the tangent becomes vertical and hence $\varphi=\pi/_2$

$$\therefore\quad \frac{1}{2}\rho gh^2=-T\Big[\cos\varphi\Big]_0^{\pi/2}=T \qquad \therefore\; T=\frac{\rho gh^2}{2}. \qquad \ldots \quad (6)$$

If θ be the angle of contact between mercury and glass, θ is the angle which the tangent to the meniscus of mercury at its point of contact B makes with glass plate through mercury. Again θ is also the angle which the tangent at the point of contact makes with the positive direction of x-axis. Integrating (5)

$$\int_0^H \rho gy\,dy=\int_0^\theta T\sin\varphi\,d\varphi$$ where H is the height of the mercury drop above the glass plate.

$\therefore\quad \frac{1}{2}\rho gH^2=T(1-\cos\theta)$......(7)

Equation (6) gives T and eliminating T from equations (6) and (7) θ is obtained.

To measure h, a short-focus microscope is mounted with its axis horizontal near the drop (fig. 97). In front of the objective a piece of plate glass, inclined at an angle of 45° to the axis of the microscope is fixed to it with a cork. A small electric lamp A is adjusted in the horizontal plane through the axis of the microscope. The rays from the lamp A traverse a convex lens L and fall on the plate glass G. Being reflected from G they fall upon the side of the drop and are again reflected from there. The height of the microscope is adjusted until the image of the filament is in coincidence with the horizontal cross-wire. The image of the filament is now evidently formed by direct reflection from the vertical portion of the periphery of the drop. The position of the microscope is noted. It is then raised and focussed on some lycopodium powder sprinkled on the upper surface of the drop. The height through which the microscope is raised gives a measure of h. The surface tension of mercury is found out from the relation (6).

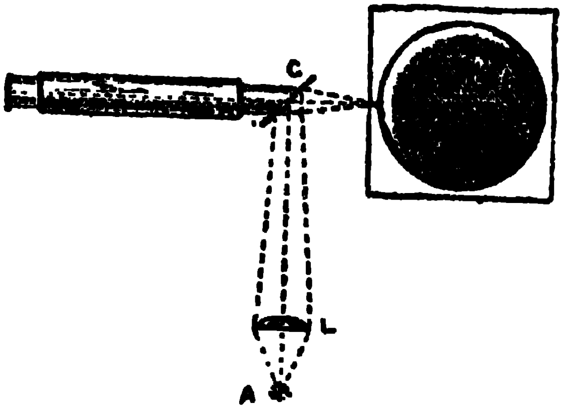

Fig. 97

The height H of the drop is measured with a large spherometer in the usual way and the angle of contact θ is calculated from relation (7).

148. Measurement of Angle of Contact—*Wilhelmy's method*—Independent methods for measurement of angle of contact are uncertain. Reliable results are, however, obtained by measuring surface tension by two methods—one. involving the angle of contact, such as capillary elevation, and the other, independent of angle of contact, such as Jaeger's method and finally calculating the angle of contact.

In Wilhelmy's method a clean glass plate is suspended vertically from one scale-pan of a common balance. It is then counterpoised by weights placed on the other scale-pan. The horizontal lower edge of the plate is brought just in contact with the liquid by lifting the container of the liquid placed on an adjustable table. Weights are then added to the other scale-pan of the balance until the plate is withdrawn from the liquid. If l be the length and t, the thickness of both sides of the lower edge of the plate and θ the angle of contact between the liquid and glass, the additional downward force on the glass plate due to surface tension$=T(l+t)\cos\theta$.

This balances the extra weight mg placed on the other scale-pan to maintain equilibrium.

$$\therefore \quad T(l+t)\cos\theta = mg$$

Measuring T by Jaeger's method, the value of θ is determined.

149. Vapour Pressure over a Curved Surface—Consider an enclosure containing a liquid and its saturated vapour only. A capillary glass tube is dipped vertically in the liquid. If the liquid moistens glass, it will be raised up the tube and its surface A within the tube will be concave upwards (Fig. 98).

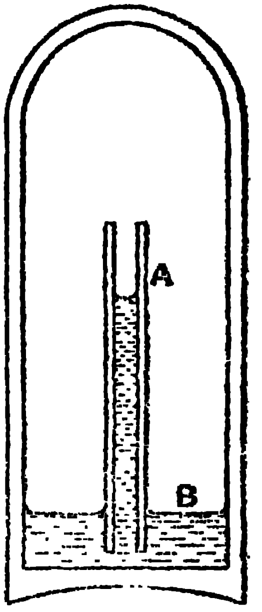

Fig. 98

Let h be the height of the liquid level A in the tube above the liquid level B (plane) in the enclosure, p_1 and p_3, the saturated vapour pressures over the concave and plane liquid surfaces at A and B and p_2 the pressure just inside the liquid in the tube below A. Let T be the surface tension of the liquid and r, the radius of the capillary tube, then r is also the radius of curvature of liquid meniscus if the angle of contact is zero. We have from eqn. (5) of Art. 145 (as $r_1=r_2=r$)

$$p_1 - p_2 = \frac{2T}{r} \text{or, } p_1 - p_2 + \frac{2T}{r} \quad \ldots \quad (1)$$

Also calculating p_3 from vapour and liquid

$$p_3 = p_1 + h\sigma g \quad \ldots \quad (2)$$

where σ=density of saturated vapour and

$$p_3 = p_2 + h\rho g \quad \ldots \quad (3)$$

where ρ=density of the liquid.

Subtracting (1) from (3)

$$p_3 - p_1 = h\rho g - \frac{2T}{r}$$

$$=(p_3 - p_1)\frac{\rho}{\sigma} - \frac{2T}{r}, \left(\text{as } hg = \frac{p_3 - p_1}{\sigma} \ldots \text{from } 2\right)$$

$$\therefore \quad (p_3 - p_1)\left(\frac{\rho}{\sigma} - 1\right) = \frac{2T}{r}$$

$$\text{or,} \quad p_3 - p_1 = \frac{2T}{r} \cdot \frac{\sigma}{\rho - \sigma}$$

since $\rho > \sigma$, the right hand side of the above equation is positive, $\therefore \; p_3 > p_1$. Hence, the vapour pressure on a concave surface is less than that on a plane surface by $\frac{2T}{r} \cdot \frac{\sigma}{\rho - \sigma}$. Similarly, the vapour pressure on a convex surface is greater than that on a plane surface by the same amount.

150. Molecular Theory of Surface Tension—Laplace's investigations, on the surface tension of liquids from the molecular conception of matter, throw a great deal of light on the subject and explain many phenomena observed in the behaviour of liquids, *viz.*, latent heat of vaporisation of liquids, etc.

He started with the assumption that the force between two molecules of a liquid is very intense when the distance between them is small. But the force falls so rapidly with increase of distance between the molecules that it may be taken as vanishingly small when the distance between the molecules exceeds a certain value c called the *range of molecular attraction.* A spherical surface imagined to be drawn round a molecule with radius c is called the *sphere of attraction* of the molecule.

When a molecule of a liquid is situated well within the interior of the liquid, its sphere of attraction lies wholly within the liquid. The molecule is attracted by finite forces in all directions equally so that there is no resultant force on the molecule to drag it in any particular direction. Its direction of motion is determined by impact with other molecules.

Near the surface, at a distance less then c (the range of molecular attraction) from the surface, a molecule is under different condition. There is a resultant force acting upon the molecule tending to drive it towards the interior. For, let *P* (fig. 99) represent the molecule near the surface. The sphere of attraction of *P* will have some portion above the surface *AB* of the liquid. Draw a plane *CD* parallel to the surface as far below *P* as the surface is above it. The molecules of the liquid, lying within the plane *CD* and the surface, exert no resultant force on the molecule at *P*, as the forces exerted by the molecules in this region on the molecule at *P* mutually counter-balance. The liquid molecules in the portion (deeply shaded) of the sphere below the plane *CD* will exert a resultant force on *P* vertically downwards which is unbalanced. For this fact, a molecule near the surface of a liquid within the range of molecular attraction from it possesses greater potential energy than when it is within the interior of the liquid, as work must be done in bringing the molecule near to the surface against this resultant force. But any mechanical system tends to come to a state in which its potential energy is a minimum. Hence, the surface area of the liquid tends to contract. The surface is thus endowed with the property of a stretched elastic membrane which is known as surface tension.

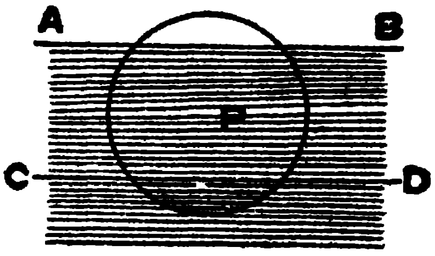

Fig. 99

In the gaseous state, the molecules of a substance are widely separated from one another and one molecule comes within the sphere of attraction of another only for a short time preceding and succeeding an impact between the two. Thus, there is no resultant force on a molecule which explains the absence of surface tension in the gaseous state.

When a molecule of a liquid in the interior gains energy, (which may be supplied to it in the form of heat) and passes into vapour, evidently a force acts upon it until it travels a distance exceeding the range of molecular attraction from the surface. The external heat-energy, which one gramme of the liquid must be supplied with, in order that it may move from within the interior to a space beyond the range of molecular attraction from the surface is the latent heat of vaporisation of the liquid.

The latent heat of vaporisation of a liquid decreases with increase of temperature. The connection of latent heat with surface tension leads us to conclude that the surface tension and latent heat of vaporisation of a liquid will decrease with increase of temperature of the liquid until both vanish at the critical temperature of the liquid.

151. Some Phenomena due to Surface Tension—

Floating Needle—If an ordinary sewing-needle is carefully placed on the surface of water, it is found to float. The experiment can be made by placing a needle on a piece of blotting-paper and floating the paper on water. The paper becomes wet and sinks while the needle remains floating. The vertical upward force on the needle arises due to (1) surface tension and (2) buoyancy of water as the surface of water is partly depressed. The needle is supported on the surface of water when the weight of the needle is balanced by the total upward force on it.*

Movement of Camphor on Water—If a piece of camphor is scraped and the small shavings are dropped on a clear water surface, they are found to move about with great vigour. The reason is that as camphor dissolves in water the solution has a smaller surface tension than pure water Probably due to the irregularity in shape of the shavings and the local condition of the water, the camphor dissolves a little faster at one side than at the other. The surface tension of the water in contact with the former side is more reduced than that of the water in contact with the latter side. The greater surface tension of the water on the latter side pulls the fragment.

Calming of Waves by Oil—By pouring oil anywhere on a vast sheet of water, such as the surface of sea, the troubles due to wind can be avoided to some extent. Due to the wind the surface film containing oil is carried forward leaving behind a comparatively clean sheet of water. The oil gathers in the forward direction and decreases the surface tension in that direction. Thus behind the wind, the surface tension being greater, the pull back is greater than the pull forward. The forward motion of the water is thus retarded.

Attraction and Repulsion of small floating Bodies—Small floating bodies, such as straw or match stems, are found to attract each other when they are floating near to one another. This takes place when all the bodies are wet by the liquid. Attraction also takes place when none of the bodies is wet. If. of two bodies, one is wet and the other is not wet, they repel each other.

When two bodies are wet by the liquid and they are sufficiently near to each other the liquid rises between them as in capillary phenomena. The pressure, at any point within the elevated portion of the liquid, falls short of the atmospheric pressure outside by the pressure due to a column of the liquid of height equal to the height of the point above the surface of the liquid outside. Thus, the bodies are pushed towards each other with a larger force due to the atmospheric pressure outside than they are pushed away from each other by the force due to the pressure inside. This results in an attraction between them.

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*For a detailed mathematical analysis of the subject consult Newman and Searle's General Properties of Matter, Art. 86.

When both the bodies are not wet by the liquid and they are sufficiently close to one another, the liquid level between them is depressed. At any point near the depressed liquid-surface between the two bodies, the pressure is less than that on the same horizontal level outside, because the former is the atmospheric pressure and the latter equals the atmospheric pressure together with a pressure due to a column of the liquid of depth equal to that of the point below the free surface of the liquid. Thus, the bodies are brought to each other by the excess of hydrostatic pressure acting outside.

When, however, one body is wet and the other is not wet, there will be a resulting force tending to separate them when they are sufficiently near to each other. The shape of the liquid meniscus near the wet body is concave upwards and near the other it is convex upwards. If the distance between the two bodies is large, the shape of the meniscus where it passes from a concave to a convex shape is horizontal. In such a case, the force due to surface tension of the liquid between the bodies tending to bring them together is equal to the force due to surface tension of the outside liquid tending to separate them and neither attraction nor repulsion ensues. If the bodies are sufficiently close to one another, the meniscus, as it passes the original undisturbed horizontal surface of the liquid, is inclined at a certain angle to the horizontal. The force attracting the two bodies together is due to the component of the surface tension resolved in the horizontal direction while the force separating them due to the surface tension of the outside liquid remains unaltered. The former being less than the latter, there takes place a repulsion between the bodies.

152. Surface Tension and Temperature—

Eotvos suggested that the surface tension of a liquid at a temperature t is directly proportional to $t_c - t$ where t_c is the critical temperature of the liquid.

Vander waals and Ferguson suggested a relation which shows that at the critical temperature at which the liquid and its vapour disappears, the surface tension of the liquid vanishes.

Ramsay and Shields gave a relation which is a modification of Eotvos's suggestion. The relation is

$$S(Mvx)^{3/2}=K(T_c-T-d)$$

where S is the surface tension at temp. T°Absol., M, the molecular weight and v, the sp. vol. of the liquid, x is known as the coefficient of associarion of the liquid at T° and K and d are constants. Tc the critical temperature on absolute scale. This expression shows that $S=0$ at a temp. $Tc-d$ *i.e.* at a temp. a little lower than the critical temp. This has been proved by Callender's expt. on water. Callender shows that the surface tension of water vanishes at 647° Absol., the critical temp. of water being 653° Absolute.

Examples

1. Calculate the amount of energy evolved when eight droplets of water (surface tension 72 dynes per cm.) of radius $\frac{1}{2}$ mm. each combine into one. (*P. U. 1949*).

Total surface energy of the 8 droplets$=8\times 4\pi(\cdot 05)^2\times 72$ ergs.

Let r be the radius in cm. of the droplet in which the 8 droplets combine. Since the volume of water remains the same

$$\tfrac{4}{3}\pi r^3=8\times\tfrac{4}{3}\pi(\cdot 05)^3 \quad \text{or,} \quad r^3=(2\times\cdot 05)^3 \quad \text{or,} \quad r=\cdot 1 \text{ cm.}$$

Energy of the single droplet$=4\pi(\cdot 1)^2\times 72$ ergs.

$$\text{Energy evolved}=8\times 4\pi(\cdot 05)^2\times 72-4\pi(\cdot 1)^2\times 72$$

$$=4\pi\times 72\times\cdot 01 \text{ or } 9\cdot 05 \text{ ergs.}$$

2. Two soap bubbles of radii 2 and 3 cm. coalesce into a single bubble of radius R cm. If the surface tension of soap solution is 25 dynes per cm. and the atmospheric pressure is 76 cm. of mercury (whose density is 13·6 gm. per c.c.) find the equation to determine R. (*M. U. 1949*).

Let r_1 and r_2 be the radii of the two bubbles coalescing into one of radius R. The mass of air in the two bubbles is evidently the same as that in the coalesced bubble.

$\therefore \tfrac{4}{3}\pi\rho R^3=\tfrac{4}{3}\pi(\rho_1 r_1{}^3+\rho_2 r_2{}^3)$ where ρ, ρ_1, ρ_2 refer to the densities of the air enclosed in the single and two separate bubbles

Since density $\propto$ pressure at constant temperature

$\tfrac{4}{3}\pi PR^3=\tfrac{4}{3}\pi(p_1 r_1{}^3+p_2 r_2{}^3)$ (P, p_1, p_2 refer to corresponding pressures)

or, $PR^3=p_1 r_1{}^3+p_2 r_2{}^3$

Again, $P=A+\frac{4T}{R}$, $p_1=A+\frac{4T}{r_1}$, $p_2=A+\frac{4T}{r_2}$

where A=atmospheric pressure, T=Surface tension

$$\therefore \quad \left(A+\frac{4T}{R}\right)R^3=\left(A+\frac{4T}{r_1}\right)r_1^3+\left(A+\frac{4T}{r_2}\right)r_2^3$$

or, $A\{R^3-(r_1^3+r_2^3)\}+4T\{R^2-(r_1^2+r_2^2)\}=0$

In the present case $A=76\times13{\cdot}6\times981=1{\cdot}014\times10^6$ dynes/sq.cm.

$r_1=2$ cm. $r_2=3$ cm, and $4T=100$ dynes/cm.

$$\therefore \quad 1{\cdot}014\times10^4(R^3-35)+(R^2-13)=0$$

This equation determines the value of R.

3. A drop of water 0·4 cm. in radius is split up into 125 equal tiny drops. Find the increase in surface energy. The surface tension of water is 72 dynes/cm. (*P. U. 1946*).

Let r be the radius of any of the tiny drops. Since the volume of water remains unaltered

$$\tfrac{4}{3}\pi r^3\times125=\tfrac{4}{3}\pi({\cdot}4)^3 \text{ or, } 5r={\cdot}4 \quad \therefore\ r={\cdot}08 \text{ cm.}$$

The increase of surface energy

$$=4\pi({\cdot}08)^2\times125\times72-4\pi({\cdot}4)^2\times72=4\pi\times{\cdot}64\times72$$

or, 579 ergs. nearly.

4. The limbs of a capillary U tube have internal diameters of 1 mm. and 2 mm. The tube is held vertically and is partially filled with a liquid of surface tension 50 dynes/cm. Find the density of the liquid if the difference of levels in the two limbs is 1·25 cm. Assume that the angle of contact is zero.

(*B. U. 1936*)

For the rise of a liquid of density ρ in a capillary tube of radius r

$$T=\frac{rh\rho g}{2\cos\theta}=\frac{rh\rho g}{2} \text{ (if } \theta=0) \text{ or, } h=\frac{2T}{r\rho g}$$

If h_1 and h_2 be the rise of liquid in the two limbs of radii ·05 and ·1 cm. respectively

$$h_1=\frac{2T}{{\cdot}05\rho g} \text{ and } h_2=\frac{2T}{{\cdot}1\rho g}$$

$$\therefore \quad h_1-h_2=\frac{2T}{\rho g}\left(\frac{1}{{\cdot}05}-\frac{1}{{\cdot}1}\right)=\frac{2T\times10}{\rho g}$$

$$\therefore \quad \rho=\frac{2T.\times10}{(h_1-h_2)g}=\frac{1000}{1{\cdot}25\times981} \text{ or, } {\cdot}8155 \text{ gm/cc.}$$

5. A capillary tube of internal diameter 1 mm. and external diameter 5 mm. hangs vertically from the arm of a balance, the lower end of the tube being in water (angle of contact$=0$). Calculate the change in apparent weight of the tube due to surface tension (surface tension of water$=78$ dynes/cm. $g=980$ cm/sec^2)

(A. U. 1948)

The force exerted by the water within the tube (Art. 138)

$=2\pi\times{\cdot}05\times78$ dynes vertically downwards.

The force exerted by the water outside the tube

$=2\pi\times{\cdot}25\times78$ dynes vertically downwards.

Hence, the total downward force which is the change in apparent weight of the tube due to surface tension

$$=2\pi\times{\cdot}05\times78+2\pi\times{\cdot}25\times78 \text{ dynes}=\frac{2\pi\times{\cdot}30\times78}{981}\text{ gm. wt.}$$

or, $\cdot15$ gm. wt.

6. Two equal spherical soap bubbles coalesce; if V is the consequent change in volume of the contained air and S, the change in total surface area, show that $3PV=4ST$ where T is the surface tension of the soap bubble and P, the atmospheric pressure.

(Allahabad U. 1943)

Let r_1 be the radius of either bubble and r_2 that when the two coalesce. The volume of air in the two bubbles

$=\frac{8}{3}\pi r_1^3$ and the inside pressure$=P+\frac{4T}{r_1}$

After they coalesce, the volume of the inside air

$=\frac{4}{3}\pi r_2^3$ and the pressure $=P+\frac{4T}{r_2}$

By Boyle's law $\frac{8}{3}\pi r_1^3\left(P+\frac{4T}{r_1}\right)=\frac{4}{3}\pi r_2^3\left(P+\frac{4T}{r_2}\right)$

or, $P\left(\frac{8}{3}\pi r_1^3-\frac{4}{3}\pi r_2^3\right)=\frac{4T}{3}\left(4\pi r_2^2-8\pi r_1^2\right)$

The quantities within brackets on the left and right side are respectively the changes of volume and surface area of soap bubbles

or, $PV=\frac{4T}{3}S$ or, $3PV=4ST$.

7. A small drop of water has a radius equal to r cms. Calculate the decrease in surface energy of the drop when the radius is diminished by a small quantity δ cms. due to evaporation.

Calculate the radius of the largest drop of water that can evaporate at $0^\circ C$ without heat being communicated to it. Surface energy of water at $0^\circ C = 117$ ergs/cm^2. Latent heat of vaporisation of water at $0^\circ C = 606$ calories/gm. (*C. U. 1949*)

Surface energy when the drop is of radius r

$$= 4\pi r^2 T \text{ } (T = \text{surf. tens. of water})$$

Surface energy when the radius is diminished by δ

$$= 4\pi (r - \delta)^2 T$$

Therefore decrease of surface energy

$$= 4\pi T\{r^2 - (r - \delta)^2\} = 4\pi T\left\{r^2 - r^2\left(1 - \frac{2\delta}{r}\right)\right\}$$

$$= 8\pi r T\delta \text{ (in dynamical units or ergs.)}$$

$$= \frac{8\pi r T\delta}{J} \ldots\text{(1) in thermal units } (J = \text{Joule's equivalent})$$

Also heat energy needed to evaporate a thickness δ of the drop $= 4\pi r^2 \delta \rho L \ldots (2)$ (in thermal units) where $4\pi r^2 \delta =$ volume of the layer, $\rho =$ density, $L =$ latent heat

Considering relations (1) & (2) the condition that a thickness δ of the drop can evaporate without supply of external energy is

$$\frac{8\pi r T\delta}{J} > 4\pi r^2 \delta \rho L$$

$$\text{or, } \frac{8\pi r T}{J} > 4\pi r^2 \rho L.$$

This result is independent of thickness δ of the drop.

The radius of the largest drop, that can evaporate without supply of heat from external sources is given by

$$\frac{8\pi r T}{J} = 4\pi r^2 \rho L$$

$$\text{or, } r = \frac{2T}{\rho L J} = \frac{2 \times 117}{1 \times 606 \times 4{\cdot}2 \times 10^7} \text{ or, } 9{\cdot}2 \times 10^{-9} \text{ cm.}$$

Exercise

1. What do you understand by surface tension, surface energy and angle of contact? What is capillary action? Explain why water rises in a capillary glass tube and why oil spreads over water. Describe an accurate method of measuring the surface tension of a liquid and give the theory underlying it. (*C.U.* 1945)

2. Find an expression for the height to which a liquid will rise above the general level in a capillary tube of narrow circular bore.

Explain the cause of rise and the form of the upper free surface in a capillary tube. Can the liquid level be depressed in any case ? If so, why ? (*C. U.* 1948)

3. Explain surface tension from the view of molecular structure of matter. How is it connected with the surface energy ?

A small drop of water has a radius equal to r cms. ; calculate the decrease in the surface energy of the drop when the radius is diminished by small quantity d cm. due to evaporation.

Calculate the radius of the largest drop of water that can evaporate at 0°*C*, without heat being communicated to it.

Surface energy of water at $0^\circ C=117$ ergs/cm^2.

Latent heat of evaporation of water at $0^\circ C=606$ gram. calories per gram. (*C. U.* 1949)

4. Explain the origin of surface tension in liquids. How does it vary with temperature and the state of electrification of the surface ?

Describe the method of finding surface tension of a liquid by the capillary tube method, deducing the formula used. (*C. U.* 1951)

5. Explain the relation between the surface tension and energy per unit area of a liquid film. (*C. U.* 1952)

6. Explain surface tension from the view of molecular structure of matter. How is it connected with the surface energy ? Describe an accurate method off measuring the surface tension of a liquid with full experimental details. (*C. U.* 1953)

7. Explain what is meant by surface tension, and describe a method for finding its value. Calculate the diameter of a capillary tube in which a liquid rises 2·50 cm. Surface tension of the liquid $=85{\cdot}0$ dynes per cm. and its density $=0{\cdot}85$. (*L. U.* 1947)

8. Define surface tension. Find an expression for the pressure inside a soap bubble. If the mass of gas inside a soap bubble is doubled, find a relation connecting the old and new radii. (*N. U.* 1950)

9. Calculate the excess of pressure in a bubble above the surroundings ; and describe a method, based on this, for measuring the surface tension of a liquid. (*R. U.* 1950)

10. Define surface tension. Show that the excess of pressure acting on the curved surface of a curved membrane is given by $p=2S\left(\frac{1}{r_1}+\frac{1}{r_2}\right)$ where r_1 and r_2 are the radii of curvature and S, the surface tension of the membrane. (*P. U. & E. P. U.* 1950)

11. Derive an expression for the difference in saturation pressure of water vapour over a spherical surface and a plane surface ; calculate the radius of the largest drop of water that would evaportate spontaneously. (Latent heat of vaporization of water $=540$ calories ; surface tension $=72$ dynes/cm.). (*N. U.* 1952)

12. What do you understand by surface tension ? What are its units and dimensions ? Find an expression for the excess of pressure inside a spherical soap-bubble over that outside. (*L. U.* 1949)

13. Write notes on Jaeger's method of measuring surface tension. (*B. U.* 1954)

CHAPTER XI

VISCOSITY

153. Viscosity—A perfect fluid is defined as one which cannot offer a tangential force whether at rest or in motion. Experience, however, shows that when there is a relative motion among the different layers of a fluid, forces are called into play within the fluid tending to destroy the relative motion. Thus, to maintain the uniformity of temperature, the contents of a calorimeter are kept *continually* stirring as otherwise the liquid comes to rest due to tangential forces called into play out of the relative motion among the different layers of the fluid. This property of a fluid which opposes a relative motion among its parts is called *viscosity.*

The opposing force called into a fluid when its different layers are moving relative to one another is the friction exerted by the molecules of the fluid and hence viscosity is called the *Molecular Friction* or *Internal Friction.* In a perfect fluid this internal friction is supposed to be absent and hence a perfect fluid is defined as a fluid which does not possess viscosity (Art. 124).

154. Orderly and Turbulent Motion : Critical Velocity—A fluid possesses two characteristic types of motion. In one (fig. 100) the path of a moving particle coincides with the line of motion of the fluid. This motion is called the *orderly or stream line motion.* In the other type (fig. 101) the path of a particle does not coincide with the line of motion of the fluid. This motion is called the *turbulent or disorderly motion.*

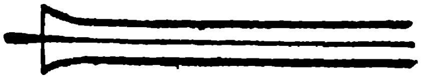

Fig. 100

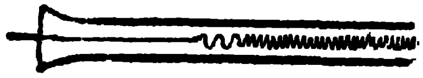

Fig. 101

Prof.. Osborne Reynolds studied the orderly and turbulent motion of water. Water was allowed to flow along a glass tube maintained at a certain difference of pressure between its ends. That end of the tube from which the water flowed through it, was provided with a small jet through which a little coloured water could be introduced along the axis of the glass tube. He found that when the difference of pressure between the ends of the tube was not high and the velocity of the water was consequently small, the coloured water flowed along the axis of the glass tube showing that the motion was orderly ; but at a certain velocity of the water, called *the critical velocity*, the band of coloured water became sinuous showing that the turbulent motion had set in.

155. Coefficient of Viscosity—Consider a stream of fluid flowing over a fixed horizontal plane AB (fig. 102) from left to right. The layer of the fluid in contact with AB is at rest while the velocities of different layers above AB are increasing uniformly according to their heights above AB. If we consider an imaginary horizontal plane CD within the fluid, the fluid molecules below CD exert a force on it from right to left as shown by the arrow-head which tends to destroy the relative motion among the different layers of the fluid. If v be the velocity of the fluid-layer CD at a height d above AB, the change of velocity per unit distance between AB and CD or the *velocity gradient* $=\frac{v}{d}$.

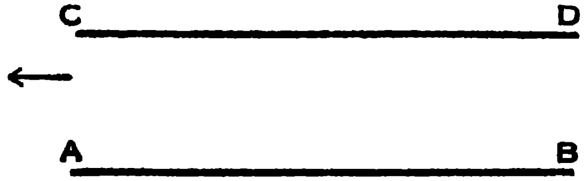

Fig. 102

Newton assumed that the tangential stress to destroy the relative motion is proportional to the velocity gradient. Thus, if F be the total force on CD of area S tending to destroy the relative motion the tangential stress F/S will be given by

$$\frac{F}{S} \propto \frac{v}{d} \text{ or, } \frac{F}{S} = \eta\frac{v}{d}$$

where η is a constant depending on the nature of the fluid. The constant η is called the *coefficient of viscosity of the fluid.*

If the velocity gradient $\frac{v}{d}$ is unity, $\eta = \frac{F}{S}$

Thus, the coefficient of viscosity may be defined as the tangential force per unit area per unit velocity gradient.

Newton's assumption is found to be true for orderly or stream-line motion of the fluid, but does not hold for turbulent or disorderly motion.

156. Fugitive Elasticity.—In a fluid, the viscosity is the tangential stress per unit velocity gradient. It has a good resemblance to the shear elasticity of a solid which is the tangential of a viscous fluid, the resistance to the motion of the body is $6\pi\eta rv$ where r is the radius of the sphere and v is the terminal velocity.* This is known as Stokes' law.

Let a sphere of radius r and density ρ fall under gravity through a medium of unlimited extent of density σ and coefficient of viscosity η.

Weight of the sphere$=\frac{4}{3}\pi r^3\rho g$

Buoyancy of the liquid$=\frac{4}{3}\pi r^3\sigma g$

Effective weight of the body$=\frac{4}{3}\pi r^3(\rho-\sigma)g$

If v be the terminal velocity attained by the body we have by Stokes' law

$$6\pi\eta rv=\frac{4}{3}\pi r^3(\rho-\sigma)g$$

$$\text{or, } \eta=\frac{2}{9}\frac{(\rho-\sigma)g}{v}r^2 \qquad \ldots \qquad \ldots \qquad (1)$$

To determine the coefficient of viscosity of a very viscous liquid, such as castor oil, the liquid is contained in a glass cylinder A (fig. 105) about 80*cm.* long and 10*cm.* in diameter. Two marks *B* and *C* are made on the surface of the cylinder such that *B* is 10*cm.* below the upper surface and *C* is 10*cm.* above the lower surface of the liquid. The sphere to be dropped in the liquid should have a diameter less than 2*mm.* These dimensions of the vessel and the sphere practically satisfy the conditions laid down in Stokes' law—that the liquid is of infinite extent.

The sphere previously moistened with the oil is gently dropped from a watch glass on the surface of the liquid along the axis of the cylinder. The interval of time t between the transits of the lower surface of the sphere between the marks is measured and the distance l between the marks is determined.

The terminal velocity $v=\frac{l}{t}$

$$\therefore \quad \text{From eqn. (1)} \quad \eta=\frac{2}{9}\frac{(\rho-\sigma)g}{l}r^2t$$

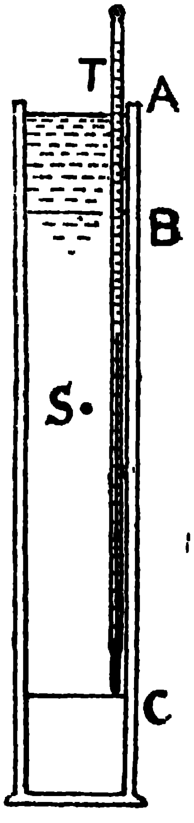

Fig. 105

For different spheres falling between the same two marks r^2t is a constant. Several steel ball bearings of less than 2*mm.* diameter previously moistened with the liquid may be used and a curve r^2 against t^{-1} plotted. This will be a straight line. As the viscosity of oil changes rapidly with change of temperature, the temperature of the liquid should be noted with a thermometer reading ·1°*C*.

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* For a proof of this from the theory of dimensions see Chapter XV and for a complete mathematical analysis see Lamb's Hydro-dynamics.

160. Viscosity of Gases—Poiseuille's equation for the rate of flow of a liquid through a capillary tube is not directly applicable to the case of gases. For, liquids being almost incompressible, the density of the liquid remains unaltered at different cross-sections of the capillary tube. In the case of gases, however, because of the variation of pressure from point to point along the capillary tubo, the density of the gas changes. What is true for a liquid or gas is that the mass of the fluid flowing through any cross-section per sec. is constant as there is no accumulation of matter anywhere in the tube. Hence ρsv=a constant where ρ=density of the gas at a cross-section, s=area of the cross-section and v the velocity of the gas across the cross-section. Since $sv=V$, the volume of the gas crossing the cross-section per sec. and $\rho \propto P$, the pressure across the cross-section (Boyle's law Art. 136), we have, for any cross-section of the tube

$$PV = \text{a const} \ldots\ldots\ldots\ldots (1)$$

Consider a small element of the capillary tube of length dx. let dP be the difference of pressure between its ends. According to Poiseuille's equation (supposing that the density remains the same for this length)

$$V = -\frac{\pi r^4}{8\eta} \cdot \frac{dP}{dx} \qquad \ldots \qquad (2)$$

The negative sign is due to the fact that as pressure falls with increase of length, $\frac{dP}{dx}$ is negative and the $-ve$ sign makes both sides identical.

If P_1 is the pressure at the inlet end of the capillary tube and V_1 the volume of gas entering the tube per second,

$$P_1V_1 = PV = -\frac{\pi r^4}{8\eta} P \frac{dP}{dx} \text{ from equations (1) and (2)}$$

Integrating for the whole length l of the capillary tube

$$\int_0^l P_1V_1\,dx = \int_{P_1}^{P_2} -\frac{\pi r^4}{8\eta} P\,dP$$

where P_1 and P_2 are the pressures at the inlet and outlet ends of the capillary tube

$$\text{or, } P_1V_1l=\frac{(P_1^2-P_2^2)\pi r^4}{16\eta}$$

$$\text{or, } V_1=\frac{\pi r^4}{16\eta l}\cdot\frac{P_1^2-P_2^2}{P_1} \qquad (3)$$

This is Poiseuille's equation for a gas.

161. Determination of the Viscosity of Air—Poiseuille's equation is applied for finding the viscosity of air in the following way : A tube ABC of moderate cross-section a (about 3 mm. in diam.) is connected to a capillary tube CD of radius r (Fig. 106). Air is forced through the capillary tube by the fall of a mercury pellet of weight mg when the tube is held vertically. The time t taken by the lower surface of the mercury pellet to fall between A and B is noted. Let Q be the volume of the tube between the marks A and B. Then the rate of entering air at the inlet end of the capillary tube is Q/t. The pressure at the inlet end is $B+b-\delta$ where B is barometric pressure, $b=\frac{mg}{\pi a^2}$ the pressure due to mercury pellet and δ is the reduction of pressure due to friction between mercury and glass wall of the tube, called the sticking coefficient. The pressure at the outlet end is the barometric pressure B. By the equation (3) of the last article.

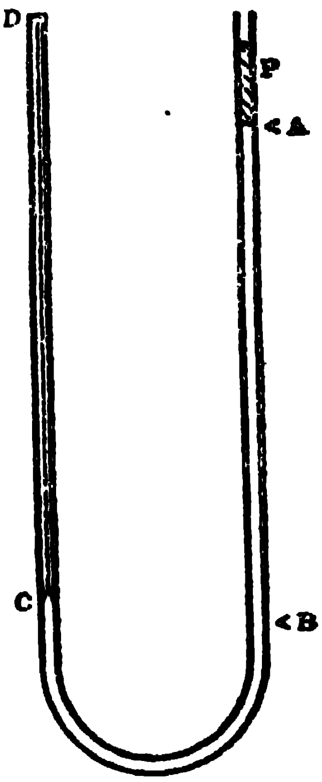

Fig. 106

$$\frac{Q}{t}=\frac{\pi r^4}{16\eta l}\cdot\frac{(B+b-\delta)^2-B^2}{B+b-\delta}=\frac{\pi r}{16\eta l}\left\{B+b-\delta-\frac{}{1+\frac{-\delta}{B}}\right\}$$

$$=\frac{\pi r^4}{16\eta l}\left\{B+b-\delta-B\left(1+\frac{b-\delta}{B}\right)^{-1}\right\}$$

$$\frac{\pi r^4}{16\eta l}\left\{B+b-\delta-B+b-\delta-\frac{b^2}{B}\right\} \text{ neglecting terms in } \frac{}{B}$$

$$Q=\frac{\pi r^4 t}{8\eta l}\left(b-\delta-\frac{b^2}{2B}\right) \quad \dots \quad \dots \quad (1)$$

Evidently $t\left(b-\delta-\frac{b^2}{2B}\right)$ is a constant.

To find δ, the experiment is conducted with two pellets of mercury ; let b_1 and b_2 be the corresponding pressures due to the two pellets and t_1 and t_2 are the respective times of fall

$$t_1\left(b_1-\delta-\frac{b_1^2}{2B}\right)=t_2\left(b_2-\delta-\frac{b_2^2}{2B}\right) \quad \dots \quad \dots \quad (2)$$

δ is calculated from equation (2) and the value of η is determined from equation (1).

162. Fluid in Motion, Bernoulli's Theorem

162. Fluid in Motion, Bernoulli's Theorem—The stream-line motion of a fluid has been defined in Art. 154. In a tube of uniform cross-section all the stream-lines *i.e.*, actual paths of the particles in the moving fluid are parallel to the axis of the tube. In general, the stream-lines are curved and the tangent at any point of it gives the direction of motion of a particle at that point. *A tube of flow* has its boundary formed by contiguous stream-lines. Bernoulli's theorem determines the manner in which pressure varies along a stream-line. Qualitatively it states that where the velocity of a fluid is high the pressure is low and where the velocity is low the pressure is high. It forms the basis of many interesting phenomena contrary to common notions:

A liquid flowing through a tube due to an external pressure possesses three kinds of energy—

(1) The work done by the external pressure in pushing the liquid through any section of the tube imparts to it an energy which is called its pressure energy.

(2) The liquid possesses kinetic energy due to its motion and

(3) If the tube is not horizontal, the liquid flowing through different cross-sections of the tube will possess different potential energy due to gravity. Bernoulli's theorem states that the total energy of a liquid flowing in stream line motion without friction from one point to another remains the same.

To deduce Bernoulli's theorem, consider a tube of flow of which the cross-sections at AB and CD are s_1 and s_2. A fluid is in stream-line motion through it in which the pressures across AB and CD are p_1 and p_2. The velocity of flow across these sections are v_1 and v_2 respectively. Also if the axis of the tube is not horizontal, the fluids crossing AB and CD per second will possess different potential energy due to gravity. Let the

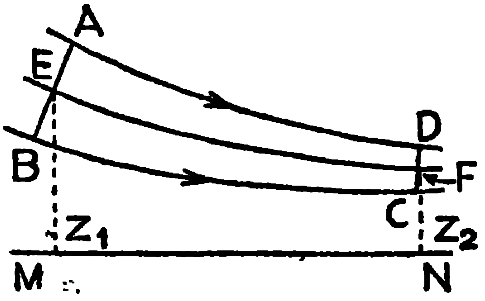

Fig. 107

heights of the axis *EF* of the flow tube above a datum or reference level *MN* be z_1 and z_2 at *E* and *F* respectively.

The work done on the liquid entering through *AB* per sec. $=p_1\ s_1\ v_1$ (s_1v_1 is the volume of liquid entering per sec.).

Kinetic energy of the liquid crossing *AB* per sec.
$=\frac{1}{2}\rho s_1v_1.v_1{}^2$($\rho s_1v_1=$mass of the liquid entering per sec.)

The gravitational potential energy of the liquid entering through *AB* per sec.$=\rho s_1v_1gz_1$

Thus, the energy entering the flow tube *ABCD* per sec. through *AB*

$$=p_1s_1v_1+\tfrac{1}{2}\rho s_1v_1v_1{}^2+\rho s_1v_1gz_1$$

Similarly, the energy leaving the flow tube *ABCD* per. sec. through *CD*

$$=p_2s_2v_2+\tfrac{1}{2}\rho s_2v_2v_2{}^2+\rho s_2v_2gz_2$$

As the energy content of *ABCD* remains unaltered,

$$p_1s_1v_1+\tfrac{1}{2}\rho s_1v_1v_1{}^2+\rho s_1v_1gz_1 =p_2s_2v_2+\tfrac{1}{2}\rho s_2v_2v_2{}^2+\rho s_2v_2gz_2 \quad \ldots \quad (1)$$

Since there is no accumulation of mass within *ABCD*

$$\rho s_1v_1=\rho s_2v_2 \quad \text{or,} \quad s_1v_1=s_2v_2 \quad \ldots \quad (2)$$

$\therefore$ from (1) and (2), $p_1+\frac{1}{2}\rho v_1{}^2+\rho gz_1=p_2+\frac{1}{2}\rho v_2{}^2+\rho gz_2$

Or, along a stream-line

$$p+\tfrac{1}{2}\rho v^2+\rho gz=\text{a constant.}$$

This is Bernoulli's theorem. It is sometimes put in the form

$$\frac{p}{\rho g}+\frac{v^2}{2g}+z=\text{a constant.}$$

$\frac{p}{\rho g}$ in called the pressure head, $\frac{v^2}{2g}$ is called the velocity head and z is the gravitational height. Also p is called the *static* pressure and $\frac{1}{2}\rho v^2$ is called the *dynamic* pressure. Hence for the stream line flow of a liquid through a horizontal tube ($z=$a constant), the sum of static and dynamic pressures remains a constant.

163. Illustrations of Bernoulli's Theorem—(1) If two cardboards are suspended one below and the other above about three inches apart by cords and a blast of air blown through them, they are found to come together. The high velocity of air between the card-boards reduces the pressure between them and they are drawn together due to greater outside pressure.

(2) The action of a scent spray is also explained in the same way. When the rubber bulb of the scent spray is squeezed, air issues from the jet at high speed so that the pressure above the

vertical tube dipped in the liquid is reduced. The greater atmospheric pressure on the liquid surface forces the liquid up the tube and the liquid passes through a second jet in the form of a fine spray.

(3) In a very high wind the roof of a house is sometimes blown off without any damage to the rest of the structure. Due to high speed wind blowing above the roof, the pressure above the roof is reduced and when the reduction is considerable the roof is lifted away as if due to suction.

164. Applications of Bernoulli's theorem :

(1) Torricelli's theorem ; Velocity of Efflux—

A liquid is contained in a wide vessel provided with an orifice *O* on the side near the bottom (fig. 108). Let the height of the surface *A* of the liquid above the axis of the orifice *B* be *h*. Bernoulli's theorem may be applied to determine the velocity of efflux of the liquid through the orifice, let *v* be this velocity. Since the vessel is wide and the orifice narrow, the velocity of the liquid at the surface layer is zero. The pressure on the liquid at the surface *A* and at the orifice *O* is each equal to the atmospheric pressure, so considering a tube from *A* to *O*, the liquid will possess no pressure energy at *A* or *O*. The liquid at *A* has no kinetic energy but it possesses only potential energy relative to the liquid issuing at *O*, the magnitude of this potential energy per unit mass is *gh*. So the total energy of the liquid per unit mass at *A* is *gh*.

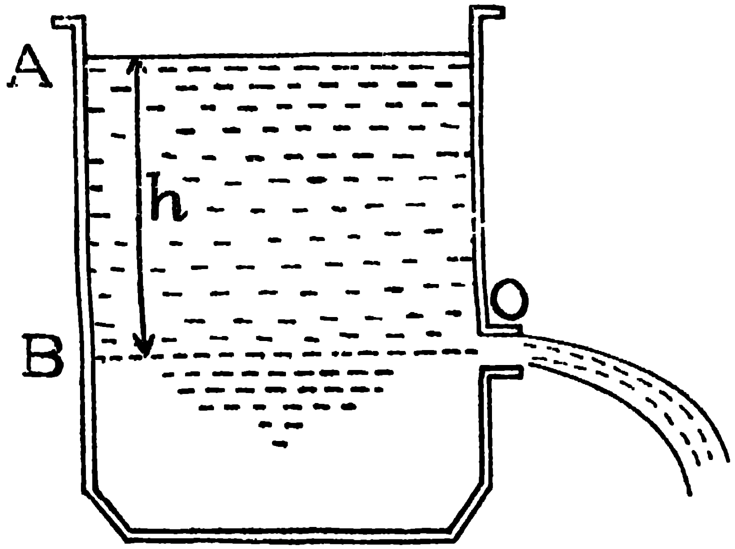

Fig. 108

Also the total energy of the liquid at O is due to its velocity and this energy per unit mass is $\frac{1}{2}v^2$

Equating the two we have

$$\tfrac{1}{2}v^2=gh \quad \text{or,} \quad v=\sqrt{2gh}$$

Thus, the velocity of efflux of the liquid through the orifice is the same as that attained by a body falling freely through a height equal to that between the surface and the orifice. In practice, however the velocity of the liquid at the orifice is less than the calculated value due to viscosity.

(2) **Venturimeter**—A practical application of Bernoulli's theorem is the venturimeter—an instrument for finding the volume of water flowing per unit time through a pipe. In its simplest form (fig. 109) it consists of two truncated conical tubes A, A connected together by a short length B of cylindrical tube called the throat. The two end tubes have cross-section equal to that of the pipe. The meter is inserted in the horizontal position in a pipe line. As water flows through the meter, its velocity will be a maximum at the throat and consequently the pressure will be reduced. The reduction in pressure is indicated by the difference H of the levels of water in the side tubes as shown in the figure.

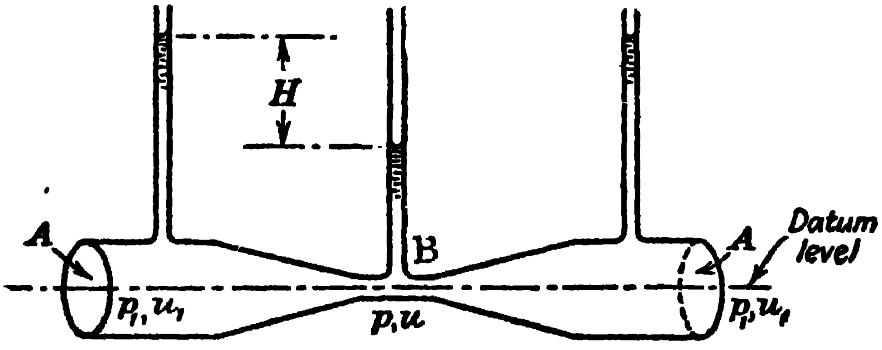

Fig. 109

Let a_1 be the cross-section of the main pipe line, a_2 that of the throat, u_1 and u_2 the velocities of water at A and B. Since the axis of the tube is horizontal there is no change of gravitational potential energy of the liquid. If the datum level passes through the axis of the meter, and p_1 and p_2 be the pressures at the wider part A and the throat B of the meter,

$$\frac{p_1}{\rho}+\tfrac{1}{2}u_1^2=\frac{p_2}{\rho}+\tfrac{1}{2}u_2^2 \qquad \text{(By Bernoulli's theorem)}$$

$$\therefore \quad \frac{p_1-p_2}{\rho}=\tfrac{1}{2}(u_2^2-u_1^2) \quad \ldots \quad \ldots \quad (1)$$

If Q be the volume of water passing any cross-section per second

$$Q = a_1 u_1 = a_2 u_2 \qquad \therefore \quad u_1 = \frac{Q}{a_1} \text{ and } u_2 = \frac{Q}{a_2}$$

Also, $p_1 - p_2 = \rho g H$

Putting these values in equation (1)

$$gH = \frac{1}{2} Q^2 \; \frac{a_1^2 - a_2^2}{a_1^2 a_2^2} \qquad \therefore \; Q = \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \sqrt{2gH}$$

165. Effect of Pressure and Temperature on the Coefficient of Viscosity—The viscosity of liquids decreases rapidly with increase of temperature. For water the viscosity at 80°*C* is only one-third of its value at 10°*C*. An empirical formula for variation of viscosity with temperature was suggested by Slotte where

$$\eta_t = \frac{\eta_0}{1 + \alpha t + \beta t^2} \text{ where } \alpha \text{ and } \beta \text{ are constants.}$$

This formula is only in approximate agreement with experiment. Andrade worked on the basis that liquid molecules vibrate about equilibrium positions which instead of being fixed as in a solid are slowly displaced with time. On this basis he deduced the formula

$$\eta v^{\frac{1}{3}} = A^{\frac{c}{vT}}$$

where v is the specific volume, A and c are constants and T, the absolute temperature. This formula agrees closely with experiment for all liquids so far examined except for water and certain tertiary alcohols.

Maxwell showed in 1860 on the basis of kinetic theory of gases, that the coefficient of viscosity of a gas would be independent of pressure and would vary directly as the square-root of absolute temperature of the gas. The first relation is found experimentally to be true except for very low pressures.

Examples

1. Water is conveyed through a horizontal tube 8 cm. in diameter and 4 kilometres in length at the rate of 20 litres per sec. Assuming only viscous resistance, calculate the pressure required to maintain the flow. (Coefficient of viscosity of water is 0·01 C. G. S. units) (Banaras U. 1949)

From Poiseuille's equation $\eta = \frac{\pi p r^4}{8lV}$ or, $p = \frac{8lV\eta}{\pi r^4}$

Here $r=4$ cm., $l=4\times10^5$ cm., $V=2\times10^4$ cc. per sec.

$$p=\frac{8\times4\times10^5\times{\cdot}01\times2\times10^4}{(4)^4.\pi}\text{ or }\frac{10^7}{4\pi}\text{ or }8\times10^5\text{ dynes per sq.cm.}$$

2. In an experiment with Poiseuille's apparatus the following figures were obtained :

Volume of water issuing per minute $=7{\cdot}08$ cc.
Head of water $=34{\cdot}1$ cm. Length of the tube $=56{\cdot}45$ cm.
Radius of the tube $={\cdot}0514$ cm.
Find the viscosity of water *(C. U. 1943)*
Pressure applied $=34{\cdot}1\times1\times981$ dynes/sq.cm.

$$\text{Volume issuing per sec}=\frac{7{\cdot}08}{60}\text{ cc.}$$

$$\eta=\frac{\pi\times34{\cdot}1\times1\times981\times({\cdot}0514)^4\times60}{8\times56{\cdot}45\times7{\cdot}08}$$

$\log\pi={\cdot}4969$	$\log 8={\cdot}9031$	log num $=1{\cdot}6436$
$\log 34{\cdot}1=1{\cdot}5328$	$\log 56{\cdot}45=1{\cdot}7517$	log deno $=3{\cdot}5048$
$\log 981=2{\cdot}9917$	$\log 7{\cdot}08={\cdot}8500$	$\log\eta=\bar{2}{\cdot}1388$
$4\log{\cdot}0514=\bar{6}{\cdot}8440$	$3{\cdot}5048$	$\eta={\cdot}01377$
$\log 60=1{\cdot}7782$		*C. G. S.* units.
$1{\cdot}6436$		

3. A vessel of cross-section 20 sq. cm. has at the bottom a horizontal capillary tube of length 10 cm. and internal radius 0·5 mm. It is initially filled with water to a height of 20 cm. above the capillary tube. Find the time taken by the vessel to empty one-half of its contents, given that the viscosity of water is 0·01 *C. G. S.* units. *(M. U. 1947)*

The internal pressure on the water varies from 20 cm. of water to 10 cm. of water. Hence the average pressure under which the water flows $=\frac{20+10}{2}$ or, 15 cms. of water $=15\times1\times980$ dynes/sq. cm.

Let t secs. be the time required. Hence the volume of water flowing out in t secs. $=20\times10$ or, 200 c.c. ∴ the rate of flow $=\frac{200}{t}$ cc. per sec. From Poiseuille's eqn. $\eta=\frac{\pi pr^4}{8lV}$

$${\cdot}01=\frac{\pi\times15\times980\times({\cdot}05)^4t}{8\times10\times200}\text{ or }t=\frac{160}{\pi\times15\times980\times({\cdot}05)^4}\text{ sec.}$$

$$=9\ \textit{min.}\ 15\ \textit{sec.}$$

4. A tank containing water has an orifice in one vertical side, the centre of the orifice is 9 ft. below the surface level in the

tank. Find the velocity of discharge assuming that there is no wastage of energy (*P. U. & E. P. U. 1952*)

From Torricelli's theorem, if v be the velocity of discharge

$$v = \sqrt{2gh} = \sqrt{2 \times 32 \times 9} = 24 \text{ ft. per sec.}$$

Exercise

1. What do you understand by a viscous fluid ?

Are the laws of viscosity in any way similar to the laws of friction ?

Compare the laws of statical, sliding and rolling frictions as far as possible. [*C. U. 1913*]

2. Define the term viscosity. Describe a method of determining it in the case of a liquid. Write down the formula you would use and give full experimental details. [*C. U. 1920*]

3. Write a short essay on the viscosity of liquids. [*C. U. 1929*]

4. A flat disc is drawn through a liquid with its surface (*a*) perpendicular, and (*b*) parallel to its direction of motion. Discuss the nature of the resistances offered to the motion of the disc in the two cases.

Describe a method of measuring the coefficient of viscosity of a liquid. [*C. U. 1931*]

5. Define the coefficient of viscosity of a liquid and find its dimensions. Describe the way in which the different parts of a viscous liquid move when flowing slowly through a fine tube What change takes place if the motion is increased ?

In an experiment with Poiseuille's apparatus the following figures were obtained :—

Volume of water issuing per minute = 7·08 c.c.
Head of water = 84·1 cm., length of tube = 58·45 cm.
Radius to tube = ·0514 cm.

Find the co-efficient of viscosity. [*C. U. 1943*]

6. Describe the way in which different parts of a viscous liquid move, when flowing slowly through a fine tube. What change takes place if the motion is increased ? [*C. U. 1952*]

7. What do you understand by the viscosity of a liquid. Give its dimensions in *C. G. S.* units. [*C. U. 1955*]

8. Define coefficient of viscosity and deduce an expression to measure it for a liquid. Deduce the formula you use. [*A. U. 1951*]

9. Define coefficient of viscosity Derive Poiseuille's formula for the volume of a liquid flowing through a tube per second. What are the limitations of this formula ? [*A. U. 1949*]

10. Water is escaping from a cistern by way of a horizontal capillary tube 10 cms. long and 0·4 mm. diameter at a distance of 50 cms. below the surface of water in the cistern. Calculate how many cubic centimetres of water will be escaping every minute assuming the acceleration due to gravity and viscosity of water to be 980 and ·01 *C. G. S.* units respectively. Prove the formula used. [*R. U. 1951*]

11. Establish Bernoulli's theorem. A tank containing water has an orifice in one vertical side. If the centre of the orifice is 9 ft. below the surface level in the tank, find the velocity of discharge assuming that there is no wastage of energy. [*P. U. & E. P. U. 1952*]

12. Deduce an expression for the rate of flow of a liquid through a tube of narrow uniform bore. Explain clearly the limitations in using the rate of flow method for the determination of the co-efficient of viscosity of a liquid. [*N. U. 1951-1954*]

CHAPTER XII

DIFFUSION AND OSMOSIS

166. Diffusion of Liquids—If two liquids, which can mix in all proportions, are placed in contact with each other, they go on mixing until a uniform mixture of the two liquids is produced. The process by which this intermixture is brought about is known as *diffusion*. The process continues in opposition to gravity. To show this, fill the lower part of a vertical glass tube with a coloured solution, such as the solution of copper sulphate. On the top of this pour clear water very slowly and carefully to avoid currents in the liquids. The blue liquid below will at first be separated by a sharply marked surface. But if the tube is left to itself, it will be found that the upper part of the tube will become gradually coloured, the colour becoming fainter and fainter towards the top of the tube while the colour in the lower part will be fainter than its original colour. After a considerable time the mixture will be nearly uniform.

167. Graham's Experiment—The first qualitative study on diffusion was made systematically by Graham (1850-1854). He took a wide-necked bottle *A* (fig. 110) and filled it with a salt solution. The bottle was closed by a glass plate and then placed in a bigger vessel *B* containing water, so that the surface of water in *B* was above the top of the bottle *A*. Communication between the salt solution in *A* and the water in *B* was established by carefully sliding the glass plate from the top of the bottle *A*, and the liquids allowed to diffuse. The arrangement was left for several days and samples of different layers of the mixed liquids were drawn off by a pipette and the composition of the solution measured. The results of Graham's experiments made on solutions of different substances and strengths can be summarised as follows :

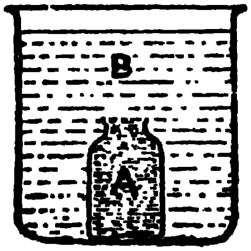

Fig. 110

The rate of diffusion of the solution of one salt was different from that of another of the same strength. The rate of diffusion of the solution of a particular salt increased with the strength of the solution proportionally. The rate of diffusion of a particular solution increased with temperature.

168. Fick's Law—The above results of Graham are embodied in a mathematical law given by Fick in 1855 and is known as *Fick's law.*

The concentration of a solution is measured by the amount (in grams.) of the solute per c.c. of the solution.

Fick's law is exactly analogous to the law for the conduction of heat. Imagine a mixture of a salt and water arranged so that the layers of equal concentration are horizontal and consider two planes of concentrations c_1 and c_2 into the mixture separated by distance x. If c_1 is greater than c_2 the quantity $\frac{c_1-c_2}{x}$ measures the change of concentration per unit distance and is called the *concentration gradient.*

Fick's law states that the mass of the salt passing per second from the plane of higher concentration to that of the lower concentration per sq. cm. is given by $Q=k\frac{c_1-c_2}{x}$. *where k is a constant depending upon the nature of the salt, the solvent and temperature, k is called the *coefficient of diffusion* or *diffusivity* of the dissolved substance.

169. Determination of the Coefficient of Diffusion of Liquids—

In a method due to Lord Kelvin (1856), the specific gravity at different layers of the solution was measured. The lower half of a vertical vessel was filled with a salt solution and the upper half contained pure water. A number of glass beads of different densities were placed in the solution. Initially some of them floated at the plane of separation of the solution and water. With the progress of diffusion, the density of the solution changed at different planes and the glass beads were displaced. By noting their positions from time to time the coefficient of diffusion was calculated. The method is objectionable owing to the fact that air bubbles are apt to stick to the surface of the glass beads and that the salt crystallises on them.

The more recent method due to Clack (1924)† consists in measuring the deviation produced in a beam of light incident horizontally on the sides of a vertical vessel containing the solution. With diffusion, the refractive index of a layer changed and this change was found out from the measurement of the change in deviation produced in the incident beam. From the change in refractive index at different layers, the distribution of concentration at different depths was obtained and the coefficient of diffusion calculated.

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*Concentration is also defined as the number of gramme-molecules of the solute dissolved in one c.c. of the solution. In that case Q will denote the number of gramme-molecules of the solute that diffuses per second through unit area from the plane of higher concentration to that of lower concentration.

† Clack *Prac. Phys. Soc.* (1924).

170. Diffusion in Gases—The phenomenon of diffusion is very rapid in gases compared to that in liquids. If a vessel is filled with two gases not uniformly mixed at the beginning, any of two gases will move from a place where its density is higher to another place where its density is lower.

A law of the same form as Fick's law for liquids is applicable to the case of gaseous diffusion. Let two gases A and B be arranged so that the layers of equal density are horizontal and let ρ be the density of a gas A at height x above a fixed horizontal plane. The *density gradient* of the gas A at any point on this plane is $\frac{d\rho}{dx}$. The law is that the mass Q of the gas A passing in unit time through unit area of this plane will be given by

$$Q=k\frac{d\rho}{dx}$$

where k is a constant depending upon the nature of the two gases A and B. It is called the *coefficient of inter-diffusion* or *inter-diffusivity* of the two gases.

171. Determination of the Coefficient of Inter-diffusion of Gases—The measurement of k, the coefficient of inter-diffusion of gases, was made by Loschmidt (1870) and by Obermayer (1880) for a considerable number of gases. A long vertical cylinder was divided into two parts by a disc in the middle. The heavier gas was placed in the lower part and the lighter gas on the upper part. The disc was subsequently removed carefully to avoid setting up of currents and replaced after a certain time. The gases in the two parts were analysed and the value of k determined.

172. Effusion or Diffusion through Fine Pores—The flow of a gas through fine pores in a thin plate is known as *effusion.* Many experiments were carried out by Graham (1846)* on effusion. The results of Graham's experiments show that if two gases A and B under the same temperature and pressure, be separated by a thin diaphragm with fine pores,

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* Phil. Trans. (1845)

$$\frac{\text{volume of } A \text{ diffused per second}}{\text{volume of } B \text{ diffused per second}} = \sqrt{\frac{\rho_B}{\rho_A}}$$

where ρ_A and ρ_B are the densities of the gases A and B respectively. In other words, the rates of effusion of different gases under similar conditions of temperature and pressure are inversely proportional to the square roots of their densities.

173. Osmosis or Diffusion through Membranes—The phenomenon of osmosis seems to have been first observed by the Abbe Nollet in 1748. He found that when a bladder filled with alcohol was tightly closed and then immersed in water the bladder began to swell out and almost burst. On the other hand, when the bladder is filled with water and then immersed in alcohol the bladder shrank.

The phenomenon is explained by the fact that water can pass through the bladder more rapidly than alcohol. The bladder thus possesses a property of selective transmission. A membrane of this type is called a *semi-permeable membrane* and the process of selective transmission is called *osmosis*.

174. Crystalloid and Colloid : Dialysis—From experiments on diffusion, Graham was led to divide substances into two classes :—Crystalloids and Colloids. The crystalloids include mineral acids and salts which can be obtained in a crystalline form. The colloids, on the other hand, include gums, albumen, starch, etc., which are amorphous and cannot be obtained in the crystalline form. Graham found that the crystalloids diffuse more rapidly than colloids. Besides this difference between the two classes, there are other important differences between them. When crystalloids are dissolved in water they diminish the vapour pressure, lower the freezing-point, and raise the boiling-point of water. Colloidal substances when dissolved in water hardly produce any such effects. They seem to be finely divided particles held in suspension in a liquid. The behaviour of the colloids in an electric field is very interesting. The colloids in an electric field are charged, some positively and some negatively, which is shown by their movements in the electrical field.

Graham in 1854 had found that solutions of crystalloids could pass through a colloidal film, such as a bladder or parchment paper more rapidly than colloids. Graham made use of this property to separate colloids and crystalloids from one another. A mixture of crystalloids and colloids are placed in a tray, the bottom of which was made of parchment. The tray was allowed to float in a vessel containing water. The crystalloids passed through the parchment while most of the colloids were left on the tray. This process was termed *dialysis* by Graham.

175. Osmotic Pressure—A suitable semi-permeable membrane for studying osmosis is made in the following way :—A porous clay pot (similar to that used in the construction of many voltaic cells) is filled with a solution of potassium ferrocyanide and dipped into a solution of copper sulphate. The two solutions meet and interact within the pores of the pot forming an insoluble precipitate of copper ferrocyanide. The porous pot, impregnated with copper ferrocyanide in this way, acts as a semi-permeable membrane which permits water but not dissolved substances to pass through it. If the pot is filled with solution of sugar and closed tightly with a stopper provided with an open glass tube (fig. 111) and is then immersed in a vessel of pure water,—water penetrates into the porous pot and the liquid rises in the tube. The process ceases when a definite excess of pressure is produced. This pressure is called the *osmotic pressure* of the solution. Its value depends upon the strength of the solution.

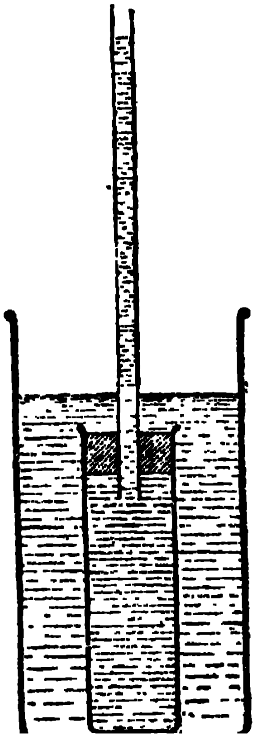

Fig. 111

The experiment is explained in the following way :—The membrane is continually receiving impacts from water molecules on one side and from the mixture of water and sugar molecules on the other side of it. Sugar is refused transmission through the membrane and consequently a greater quantity of water enters into the porous pot than what leaves it, until finally the exchange of water molecules is stopped by the hydrostatic pressure of the water in the tube.

176. Measurement of Osmotic Pressure—

(1) Pfeffer's Method—The first quantitative measurements of osmotic pressure were made by Pfeffer in 1877 with solutions of cane-sugar. His arrangement was exactly the same as described in Art. 175. When the rise of the solution in the glass tube reaches its maximum the hydrostatic pressure prevents the further penetration of water into the porous pot. The pressure of the column of the solution is equal to the osmotic pressure.

If h be the height of the column of liquid in the glass tube, s the specific gravity of the solution and g, the acceleration due to gravity, the osmotic pressure of the solution$=hsg$.

The method of Pfeffer is not sufficiently accurate, specially for concentrated solutions, when the large influx of water into the porous pot diminishes the strength of the solution. For concentrated solutions again the measurement of the high pressure involves the use of manometers. More modern arrangement of Morse (1901-1912), specially suitable for concentrated solutions, consists of the use of a membrane of copper ferrocyanide prepared in a way different from Pfeffer's. Morse obtained the membranes of much greater mechanical strength by electric endosmose.

In this process a solution of copper sulphate is taken in a porous cell, the cell is then partly immersed in a solution of potassium ferrocyanide. An electric current is sent from the copper sulphate to the potassium ferrocyanide in such a direction that the copper ions and the ferrocyanide ions travel through the porous cell where they interact and get deposited within the pores as a membrane of copper ferrocyanide. The completion of the process is indicated by an increase in electric resistance when the current almost stops. The cell is then thoroughly washed and it becomes ready for use.

(2) **Berkeley and Hartley's Method**—A novel method for measurement of osmotic pressure eliminating the defect of Pfeffer's method was introduced by Berkeley and Hartley in 1906. A horizontal porcelein tube AA (fig.112) has a semi-permeable membrane of copper ferrocyanide. B, B represents a gun-metal case enclosing AA and fixed solidly to it. The side tube D lead to a vertical graduated open capillary tube and the other side tube E is provided with a stop-cock. The solvent (water) is placed in the porcelein tube A with a funnel through E by opening the stopper

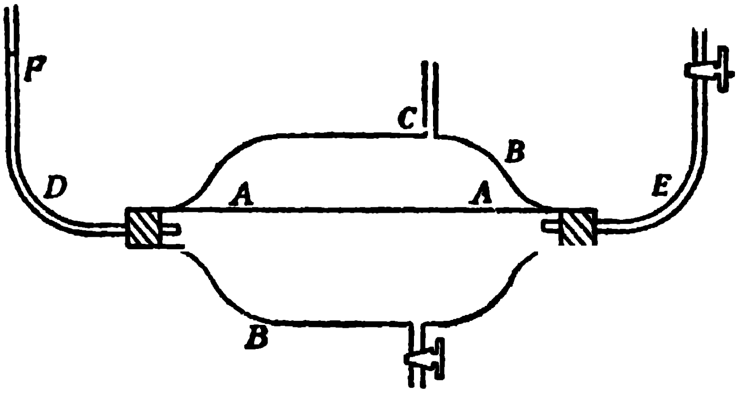

Fig. 112

until its level rises to *F* in the side tube *D*. The gun-metal tube *BB* is filled with the solution under test introduced through the side tube *C*. The solvent in *A* tends to pass through the semi-permeable membrane into *B* which is indicated by the fall of its level in the tube *D*. Pressure is externally applied through *C* to keep the level of the solvent in *D* constant. The applied pressure is evidently equal to the osmotic pressure of the solution. The method is very rapid, also the membrane is strong enough to allow osmotic-pressure of 130 atmospheres to be measured.

177. Definition of Osmotic Pressure—If a solution is separated from the solvent by means of a semi-permeable, membrane, the pressure-difference which has to be established between the solution and the pure solvent to check the inflow of the solvent into the solution and establish equilibrium in the system is called the osmotic pressure of the solution at its temperature and concentration.

Solutions having same osmotic pressure are called isotonic solutions.

178. Modern View of Osmotic Pressure—The osmotic pressure of dilute solutions is regarded as an effect similar to that of gases. The value of the osmotic pressure is calculated as the normal momentum imparted to the boundary surfaces per unit area per second by the impact of the ions of the solute. It is assumed that no force exists between the ions. Debye and Huckel's theory is an advance analogous to that of Vander Waals in the kinetic theory of gases. Their theory takes into account the forces exerted by the ions upon one another as they are electrically charged. Those forces are governed by the law of inverse square and affected by the dielectric constant of the solvent which fills the spaces between the ions. This affects the kinetic energy of the ions and reduces the osmotic pressure. In solutions of weak electrolytes the charged ions also exerts forces on one another, but as they are relatively few in number compared with the total number of undissociated molecules present, the effect is relatively small and weak electrolytes behave according to classical theory.

179. Laws of Osmosis—The quantitative relation between the osmotic pressure of solution and its concentration was discovered experimentally by Pfeffer in 1877. Three laws, known as laws of osmosis, were discovered from the experimental result of Pfeffer. These laws are true for weak solutions which do not conduct electricity. Van't Hoff who discovered the laws pointed out a striking parallelism between the properties of gases and the osmotic properties of solutions.

Law I. The osmotic pressure of a solution at constant temperature is directly proportional to the gramme-molecular concentration (number of gramme molecules per c.c.) of the solution.

$\therefore$ $P \propto c$ where P is the osmotic pressure and c is the gramme-molecular concentration. If V c. c. of the solution contains one gramme-molecule of the solute, c is given by $c=\frac{1}{V}$.

$\therefore$ $P \propto \frac{1}{V}$ or, $P=\frac{K}{V}$ where K is a constant

or, $PV=K$.

This law is analogous to Boyle's law for gases.

Law II. The osmotic pressure of a solution at constant concentration is directly proportional to the absolute temperature of the solution.

$\therefore$ $P \propto T$ when c is constant (where T is the absolute temperature of the solution).

This law is analogous to the pressure-temperature law of a gas.

Law III. Equimolecular quantities of different solutes dissolved in the same volume of a solution exert equal osmotic pressure at the same temperature.

This law is analogous to Avogadro's hypothesis for gases according to which equal volumes of all gases under same temperature and pressure contain equal number of molecules. Correspondingly for solutions, equal volumes of all solutions having the same temperature and osmotic pressure contain equal number of the molecules of the solutes dissolved.

The first two laws are,

$P \propto \frac{1}{V}$ and $P \propto T$

Combining them $P \propto \frac{T}{V}$ or, $\frac{PV}{T}=R$ (a constant).

This is the well-known equation of state for a perfect gas. The value of R is the same as for gases.

For gases, 1 gm. molecule of any gas occupies 22·4 litres at N.T.P. Hence, the value of the molar gas constant R is given by $R=\frac{PV}{T}=\frac{1 \times 22·4}{273}=·082$ litre-atmosphere per degree. Also from Pfeffer's data a solution of cane-sugar of 1 gm. in 100 c. c. or 1 grammolecule (342 gms.) in 34200 c. c. or 34·2 litres exerts an osmotic pressure of 0·649 atmospheres at 0°C.

Hence, $R=\frac{PV}{T}=\frac{\cdot649\times34\cdot2}{273}=\cdot0813$ litre-atmosphere per degree.

This is nearly the same as the gas constant R, *viz.*, ·082. Thus, osmotic pressure of a solution obeys the gas laws provided the solution is non-electrolyte and dilute *i.e.*, the volume of the solute is negligible compared to that of the solvent.

Van't Hoff in 1888 gave a proof of the above laws from thermo-dynamics. The laws are known as Van't Hoff's osmotic pressure laws. He showed that the above laws strictly hold good for solutions of infinite dilution.

180. Osmotic Pressure of Electrolytes—Although Van't Hoff's laws of osmotic pressure are obeyed by the aqueous solutions of substances, such as cane-sugar, marked deviations are met with the solutions of electrolytes. The osmotic pressure with these solutions are still found to be approximately proportional to molecular concentration, but for a particular concentration the value of the osmotic pressure is considerably greater than the calculated value.

Arrhenius succeeded in explaining these deviations by his *electrolytic dissociation theory*, according to which, the molecules of an electrolyte in a solution are dissociated or broken up to a greater or less extent into their atoms. Thus, a molecule of $NaCl$ is broken up into an atom of Na and an atom of Cl. He further supposed that these atoms were electrically charged and called these charged atoms as *ions*. The Na atoms are charged positively, move *with* the current in an electric field and are liberated at the cathode and they are called *cations*, whereas the Cl atoms are charged negatively, move *against* the current in an electric field and are liberated at the anode and they are called *anions*. These *ions* move under electric force and render the solution a conductor of electricity.

The large osmotic pressure of an electrolyte is due to the fact that the molecules of the electrolyte are partially dissociated in the solution. The *ions* thus formed have their own osmotic effects. Thus, the osmotic pressure of the solution is the sum of the osmotic pressures of the two kinds of *ions* plus the osmotic pressure of the remaining undissociated molecules.

Examples

1. 10 gm. of sugar of molecular weight 360 is dissolved in 1 litre of water, the temperature being 15°*C*. If the gas constant per gm. molecule is $8{\cdot}26\times10^7$, calculate the osmotic pressure of the solution.

1 gm. mol. of sugar is contained in $\frac{360\times1000}{10}$ or 36×10^3 c.c.

$$\therefore \text{ The osmotic pressure}=\frac{RT}{V}=\frac{8{\cdot}26\times10^7\times288}{36\times10^3}$$

$$=66{\cdot}1\times10^4 \text{ dynes per sq. cm.}$$

2. A solution of 24·67 gm. of colloidal tungstic acid per litre gave an osmotic pressure of 25·2 cm. of mercury at 17°*C*. Find the molecular weight of colloidal tungstic acid. (*I. C. S. 1930*)

Let *M* be the molecular weight

Hence 1 gm. mol. of tungstic acid is contained in $\frac{M}{24{\cdot}67}$ litre of the solution.

The osmotic pressure exerted $=25{\cdot}2$ cm. of mercury $=\frac{25{\cdot}2}{76}$ atmosphere.

From eqn. $PV=RT$, $\frac{25{\cdot}2}{76}\times\frac{M}{24{\cdot}62}={\cdot}082\times(273+17)$ (as $R={\cdot}082$ litre - atmos.)

$$\therefore \quad M=\frac{76\times24{\cdot}62\times{\cdot}082\times290}{25{\cdot}2}=1769.$$

3. At 10°*C*, the osmotic pressure of an aqueous solution of urea is 500 mm. of mercury. The solution is diluted and the temperature is raised to 25°*C*, when the osmotic pressure is found to be 105·3 mm. Determine the extent of dilution. (*C. U. 1935*)

Let the volume *V* of the solution be diluted to a volume *V'*

We have from the relation $\frac{PV}{T}=$ a constant

$$\frac{500\times V}{273+10}=\frac{105{\cdot}3\times V'}{273+25} \qquad \therefore \frac{V}{V'}=\frac{105{\cdot}3\times283}{500\times298}=\frac{1}{5}$$

Hence the dilution is 1 : 5

4. Calculate the osmotic pressure of a 5 percent cane-sugar solution (mol. wt—342) at 27°C. What strength of urea (mol. wt—60) solution is isotonic with this solution.

In the cane-sugar solution 1 gm. mol. of cane-sugar is contained in $\frac{342\times100}{5\times1000}$ or $\frac{342}{50}$ litres. If P be the osmotic pressure of this solution

$$\therefore\quad \frac{P\times342}{50}={\cdot}082\,(273+27)\text{ or, }P=\frac{{\cdot}082\times300\times50}{342}$$

or, $3{\cdot}596$ atmosphere.

Let the strength of the isotonic urea solution be x per cent.

$\therefore$ 1 *gm* mol. of urea is contained in $\frac{60\times100}{x\times1000}$ or, $\frac{6}{x}$ litres.

Hence $\frac{3{\cdot}596\times6}{x}={\cdot}082(273+27)$

$$\therefore\quad x=\frac{3{\cdot}596\times6}{{\cdot}082\times300}\text{ or }{\cdot}9\text{ nearly}$$

$\therefore$ Strength of the urea solution is ·9%

5. A solution containing 3·12 gm. of a substance per litre gives an osmotic pressure of 1·27 atmospheres at 25°C. Calculate the molecular weight of the substance. (*C. U.* 1942)

Let M be the molecular weight of the substance. Therefore one gm. mol. of the substance is contained in $\frac{M}{3{\cdot}12}$ litres.

From equation $PV=RT$, $\frac{1{\cdot}27M}{3{\cdot}12}={\cdot}082\,(273+25)$

$$\therefore\quad M=\frac{3{\cdot}12\times{\cdot}082\times298}{1{\cdot}27}\text{ or }60\text{ nearly.}$$

Exercise

1. Write short notes on—surface tension, diffusion, osmosis, crystalloids and colliods.

2. Define osmotic pressure of solution and describe an arrangement for making a direct measurement of the pressure. State the laws and combine them into a single formula. [*C. U. 1933*]

8. State and explain the laws of osmotic pressure. Why is it said that the solute in dilute solution behaves like a gas ?

Describe an experiment for the direct determination of the osmotic pressure of a solution. *[C. U. 1938]*

4. What do you understand by osmosis, dialysis and diffusion ? State the laws of osmosis and describe an arrangement for measuring osmotic pressure. *[C. U. 1944]*

5. What is meant by osmosis ? Write a short account of the relation between osmotic pressure and the concentration, pointing out similarities which have been observed between this phenomenon and the property of gases. *[C. U. 1947]*

6. State and explain the laws of osmotic pressure. Why is it said that the solutes in a dilute solution behave like a gas ?

Describe an experimental arrangement for determining the osmotic pressure of a solution. *[C. U. 1949]*

7. Explain what do you mean by Osmosis and Diffusion. State the laws of Osmosis and describe an accurate arrangement for measuring Osmotic pressure. *[C. U. 1951]*

8. Write short notes on :—Diffusion and Osmosis. Give a short account of the relation between the Osmotic pressure and concentration of a solution drawing attention to any similarities which have been observed between the phenomenon and the properties of gases. *[C. U. 1953]*

9. Explain osmosis and state the laws of osmotic pressure.

[C. U. 1955]

CHAPTER XIII

PRODUCTION AND MEASUREMENT OF LOW PRESSURE

181. Rotary Pump—Dr. Gæde's Rotary box pump may be used both as an exhaust pump and as a blower. It produces a vacuum upto 0·01 mm. of mercury and as a blower produces a pressure of one atmosphere.

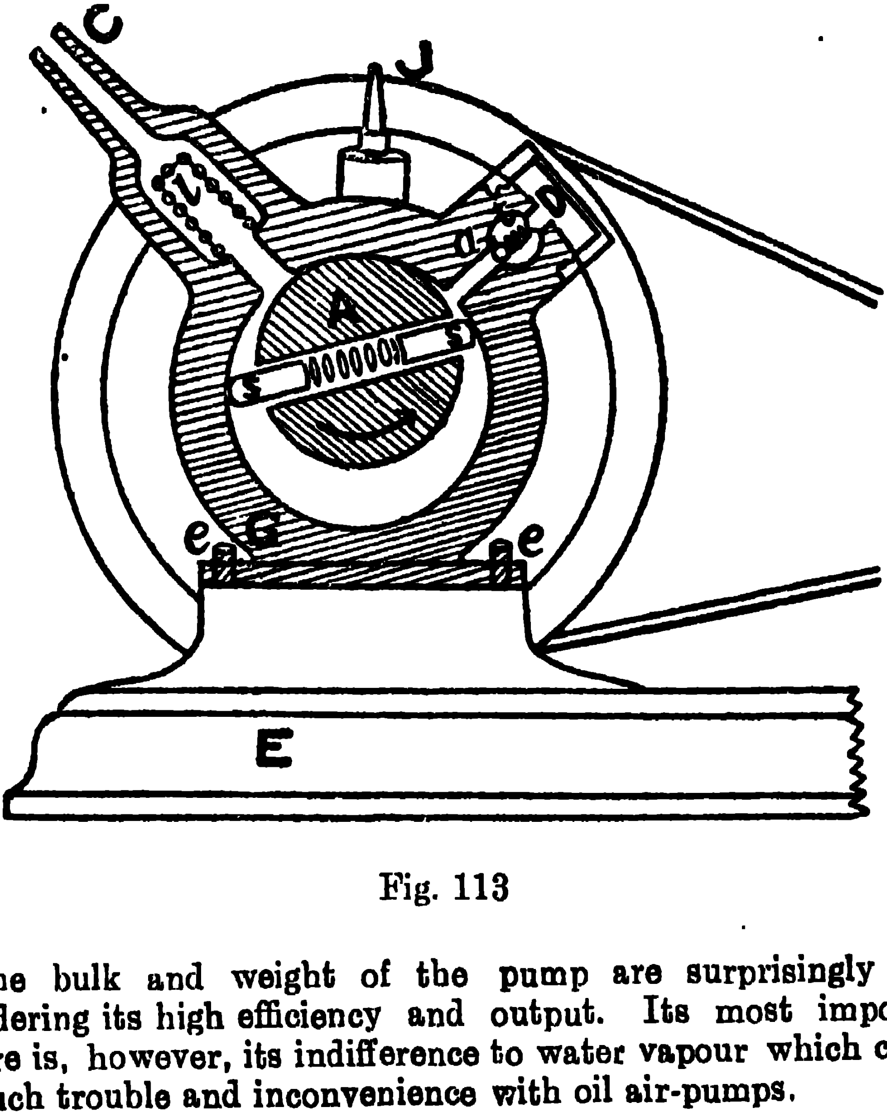

Fig. 113

The bulk and weight of the pump are surprisingly small considering its high efficiency and output. Its most important feature is, however, its indifference to water vapour which causes so much trouble and inconvenience with oil air-pumps.

Fig. 113 is a vertical section of the pump perpendicular to the shaft. The shaft carries a cylinder *A* slotted radially to receive the hardened steel valve plates *S* called 'scrapers'. They are pressed outwards against the inner surface of the red brass casing G by springs. The front of the casing G is closed air-tight (without packing) by a red brass plate. This is ground to the flange of the casing, adjusted in position and fixed firmly by several screws to the flange of the casing. The latter is mounted on an iron base E and secured by two screws e, e. A box (not shown in the figure) serves both as an oil-tank and as an air-chamber and is secured into the casing G. The tank is filled to the upper edge of a window with oil which is supplied to the shaft for lubrication, the escape of the compressed air along the shaft from within the air-chamber is prevented by a stuffing-box.

The valve D consists of the valve disc a held down on the valve seat by a spring t. The driving-pulley is fixed to the shaft and is geared to the motor by a belt, or to the hand-driving gear-wheel by a chain. The electric motor or the hand-gear-wheel is mounted with the pump on the common base E.

When the cylinder A rotates in the direction as shown in the figure by the arrow head, the air is sucked in through C and forced through the valve D and a passage into the air-chamber. From the air-chamber the air passes out through the nozzle J. The suction nozzle C contains a fine wire sieve l which retains particles of dust, fragments of india-rubber from old rubber tubes and drops of mercury carried along by the air current, so that the pump runs smoothly and needs little or no attendance. To use the pump as a blower, the nozzle J is connected to the vessel to be blown into. The pump is started by turning an ordinary lamp switch mounted on the base E. Its chief utility lies in running other high vacuum pumps which require a preliminary vacuum for working.

182. Diffusion Pump.—The process of diffusion of gases has been utilised as an efficient method for producing a high degree of vacuum in a short time. Many forms of diffusion pumps have been devised. The original form of Gæde involved complicated glass construction which is no longer used. All these pumps require a preliminary reduction of pressure to different extents depending upon the nature of the pump.

Volmer in 1919 designed two jet pumps which, if desired, may be joined in series to form a two-stage pump.

The adjoining figure (fig. 114) shows one of the Volmer pumps known as the 'downward annular jet type.' Clean pure mercury particularly free from grease is taken in the boiler A.

The vessel to be evacuated is connected at *B* preferably by fusing the glass ends. The fore-vacuum required for its working is provided by the other type of the Volmer pump or by a rotary box pump connected at *C* in the same manner. When the required fore-vacuum has been produced, the mercury in the boiler *A* is heated which is kept boiling steadily but not violently.

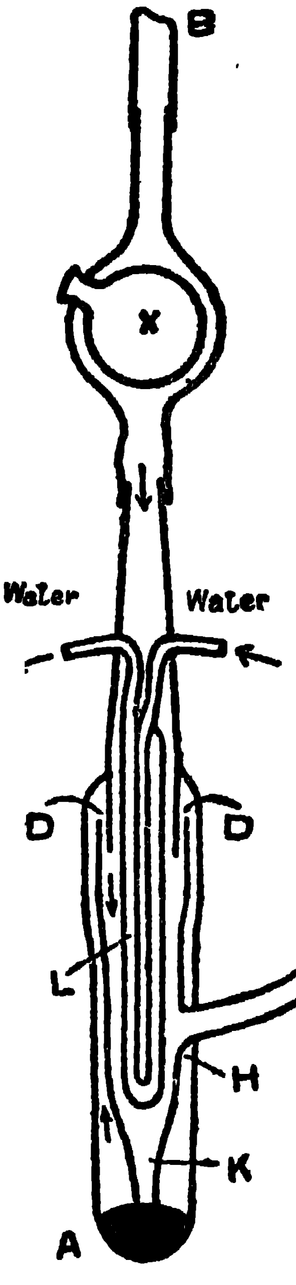

Fig. 114

The mercury vapour thus formed rises between the outer walls of the boiler and passes through the annular-jet *D*. After passing through the jet the mercury vapour meets the air molecules diffusing from the vessel. The rates of diffusion depend on their respective partial pressures. The mercury vapour comes in contact with the wall of the condenser *L*, through which water circulates, and is condensed. The condensed vapour returns to the boiler through the narrow tube *K*. The air molecules, which are entrained by the mercury vapour, are sucked in by the auxiliary pump through the tube *H*. Any mercury vapour issuing upwards is arrested in the mercury vapour trap *X*.

Gæde showed theoretically that the width of the aperture must be comparable to the mean free path (Art. 188, foot-note) of the air molecules diffusing into the mercury vapour. If the aperture is too wide a blast of heavy mercury molecules will pass through it and overpower the air molecules, which will thereby be forced back. If the aperture is too narrow the friction will be great and the diffusion consequently slow, so that the action will be stifled. The vapour pressure of mercury has a similar effect. If the vapour pressure is too small, the stream of vapours entraining the diffused gas will not be strong and the evacuation will be poor. If the vapour pressure is too great a contrary current will be created by the vapours which will hinder diffusion and render the evacuation inadequate. The conditions for successful pumping are therefore very critical.

With the help of a fore-vacuum of 10^{-1} mm., the pump produces a vacuum of the order of 10^{-6} mm. of mercury.

Gæde's measurements show that the speed of the pump is fairly low but its great theoretical advantage over other types of pumps is that its speed remains fairly constant even below a thousandth of a millimetre of mercury.

The pump is used in the manufacture of electric bulbs, thermionic valves and for running X-ray bulbs, etc., where a production of high vacuum is necessary.

183. Measurement of low Pressure :—McLeod Gauge—The well-known McLeod Gauge for the measurement of low pressure was designed by professor McLeod in 1874.

The principle of the gauge consists in compressing a known volume of the gas whose pressure is to be measured to a very small known volume. The pressure exerted by the compressed gas is observed and its original pressure is calculated by applying Boyle's law.

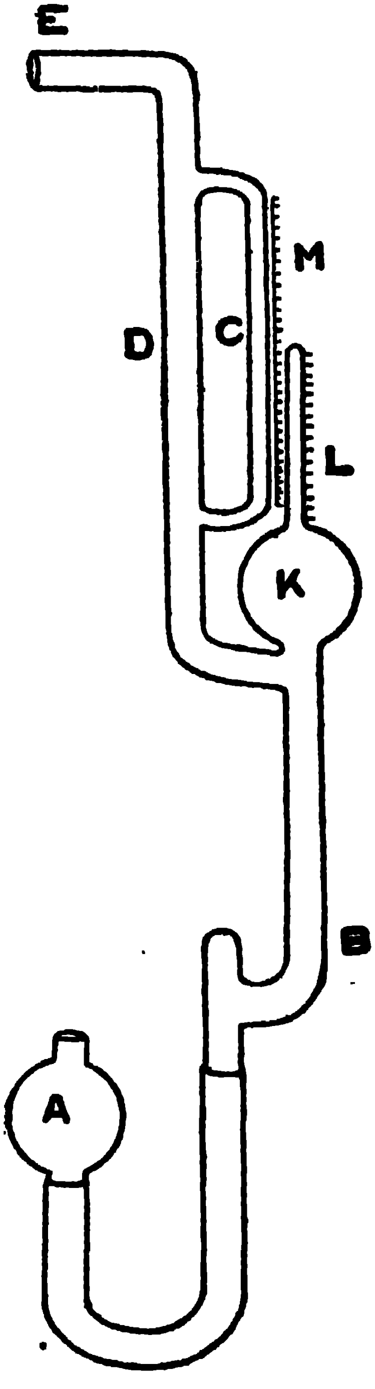

Fig. 115

Fig. 115 shows a simple form of the McLeod gauge. The bulb *K*, of which the volume is known, is attached to a closed capillary tube *L* at its top. The capillary tube *L* is graduated in c. cs. The bottom of the bulb is connected to the low pressure system at *E* through the tube *D*. *B* represents a barometric column. Another tube *C* of the same capillary bore as of *L* is sealed as a by-path to the larger tube D, to avoid errors due to the effect of capillarity. A millimetre scale M allows the measurement of the difference in mercury levels in L and C.

To measure the pressure of the evacuated system, the end E of the gauge is connected to it by fusing the glass ends. The gauge is evacuated with the system. The mercury reservoir A is then gradually raised so that mercury rises in the bulb K and the tubes B and C. The bulb K is thus shut off from the remainder of the system and the gas contained in it is compressed into the capillary tube L. The volume of the compressed gas is observed from the graduations on the capillary tube.

Let the volume of the gas compressed into the capillary tube be v, the original volume of the gas contained in the bulb and capillary tube be V, and h be the difference in the mercury levels in the capillary tubes L and C in millimetres. Then if P denotes the pressure of the gas in the low pressure system in millimetres of mercury, from Boyle's law,

$$PV=hv.$$

From this the value of P is calculated out.

Exercise

1. Write full notes on the production and measurement of low pressure. [*C. U. 1942*]

2. Describe and explain the working of a modern high vacuum pump. What is the degree of vacuum that can be created by its use? [*A. U. 1949*]

3. Describe the construction and working of mercury diffusion pump with a neat diagram. What is the order of pressure attained by such a pump? [*R. U. 1952*]

4. Describe in detail the working of the Hyvac pump and show how the pressure attained is measured. [A. U. 1946, *L. U. 1948*]

5. Write an essay on the production and measurement of low pressure. [*P. U. E. U. 1950*]

6. Give an account of the techniques employed in the production and measurement of very low pressures. [*N. U. 1955*]

7. Give a general account of high vacuum pumps and describe any one of them. [*C. U. 1956*]

CHAPTER XIV

KINETIC THEORY OF MATTER

184. Evidence of Molecular Motion—The phenomenon of diffusion exhibited by liquids and gases (Chapter XII) suggests that the molecules of a fluid are in a state of motion. It is further observed that the phenomenon of diffusion is very rapid in gases compared to that in liquids. It is also a matter of common observation that if an evacuated electric bulb is broken, air rushes in with almost explosive energy. The molecular motion of gases has been turned to a good account in the construction of the diffusion pump described in Art. 181, which is now used for the production of very high vacuum. These illustrations are suggestive of the fact that the molecules of a gas are in a state of rapid motion. Besides these illustrations, the effects of molecular motion were shown in striking experiments of Robert Brown in 1827 long before the advent of the kinetic theory.

185. Brownian Movement—The curious phenomenon discovered by Robert Brown, which goes by the name Brownian movement, can be observed in liquids and gases in the following manner :

A small quantity of an insoluble substance, such as powdered gamboge or finely ground mica, is mixed with water. The powdered solid particles are held in suspension in water and are of very different sizes. When the particles are viewed by a powerful microscope, the smaller ones are seen to be in a state of continual irregular motion as shown in fig. 116* which shows the result of three separate experiments with the same granule at intervals of 30 seconds. The smallest ones are found to move as though they were alive. The very large ones are, however, found to remain practically at rest.

The explanation of this curious effect is simple. The solid particles are receiving continual blows by the molecules of water. The solid particles are, however, enormously large in size compared to the molecules of water. Some of the solid particles have so large a surface that at every time they are receiving billions of blows on all sides and these blows perfectly balance producing no motion of the solid particles. With the smaller solid particles, however, the blows are not always symmetrically distributed, specially, if the particles are sufficiently small and at some ins-

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[*] The figure is reproduced from 'Brownian Movement and Molecular Reality.' Jean Perrin.

tant of time there may be more blows on one side than on the opposite. Hence, such solid particles will spin around or move a little in the direction of the resulting effect.

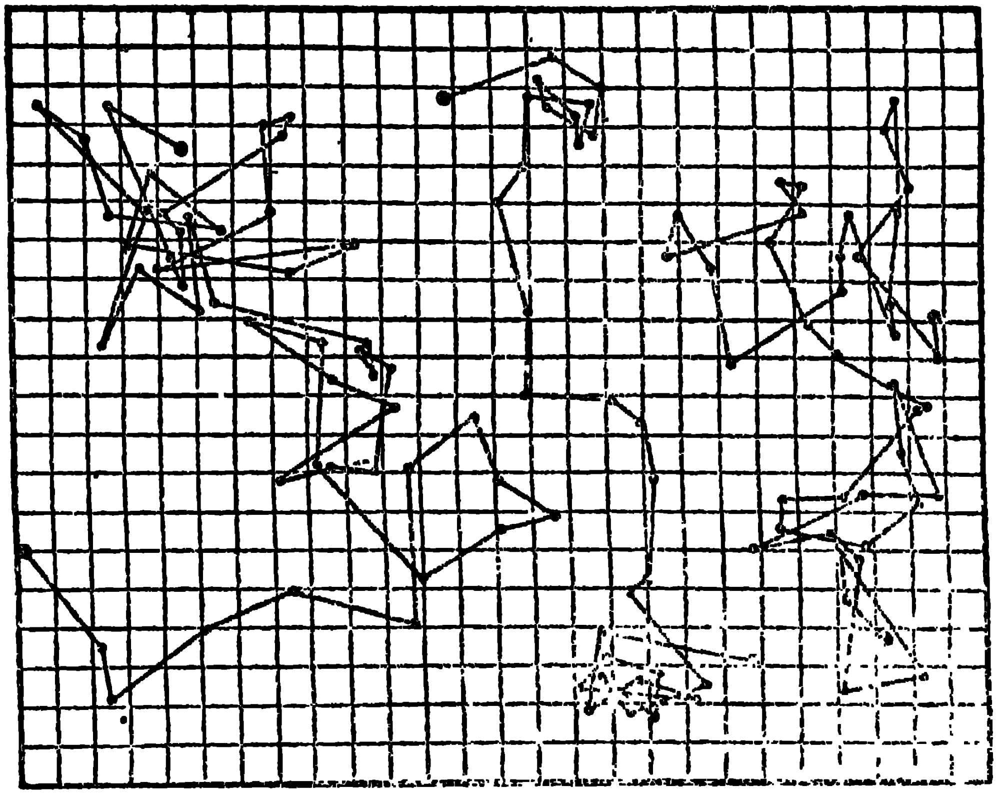

Fig. 116

To show the effect in a gas, a small shallow chamber having glass walls and containing air is put under a microscope. A little smoke is drawn into the vessel and the smoke particles are viewed by reflected light against a dark ground. The smoke particles are found to wander about the field of view and they often move out of the field of view as they are more free to move in the air than the solid particles held in suspension in water. The movement of the solid particles in water or that of the smoke particles just described is known as Brownian movement.

186. **Radiometer**—Another direct outcome of the molecular motion of gases is shown by the radiometer devised by Sir William Crooks. The radiometer (fig. 117) consists of a very light vane constructed of mica discs fastened at the ends of four arms of an aluminium wire so as to have a small moment of inertia. Each of the discs is coated with lamp-black on one side such that all the blackened sides face the same way. The

arrangement can turn very freely about a vertical axis and is enclosed in a glass vessel which is highly exhausted.

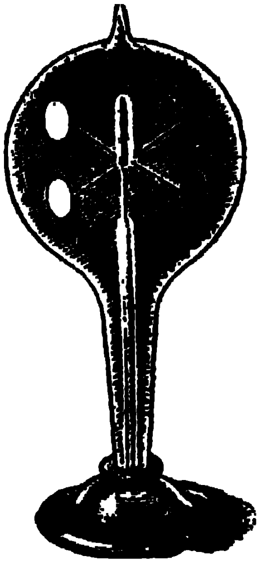

Fig. 117

If the radiometer is exposed to sunlight or to any convenient source of radiant heat, the vane revolves such that all the blackened surfaces are receding.

The explanation of the above phenomenon is given in the following way: The motion of the vane is due to the impact of the gas molecules within the radiometer on the surfaces of the discs. Lampblack being a very powerful absorbent, the sides of the discs coated with it become a little warmer than others. The gas molecules which strike on the blackened surfaces become a little warmed and rebound with a greater velocity than those which impinge on the clear sides. The pressure applied (Art. 189) on the black surface is thus greater than that on the clear surface. The vane is thus set to rotation such that the black surfaces are receding.

Dewar communicated the radiometer with a chamber containing cold powdered charcoal and made the vacuum inside as perfect as possible. The charcoal further absorbed the gas molecules and the vacuum was so perfect that the vanes no longer rotated. This is evidently due to the want of sufficient number of gas molecules within the radiometer to produce the requisite pressure by their impact needed for rotation of the vane.

187. Nature of Molecular Motion—The kinetic theory of matter, more specially that of gases, has its basis upon several phenomena and facts, such as diffusion, etc. It explains many experimental observations, such as Boyle's law, Avogadro's hypothesis, etc. It deduces a large number of conclusions capable of experimental verification and its results are far-reaching and surprising but always very near the truth.

According to the kinetic theory, the motion of the atoms and molecules of a *solid* is to a large extent very limited. The atoms and molecules of a solid are sufficiently close together so that one reacts considerably on the motion of another. Every atom or molecule is supposed to move about some mean position. The motion of an atom or molecule is thus supposed to be vibratory but not translatory although Austin has shown that a sheet of gold laid on a plane block of lead diffuses into the latter very gradually.

The molecules of a *liquid* not only move about their mean positions but they are also slowly translated by means of diffusion. The molecules are also supposed to be close to one another and the motion of one molecule is reacted by another.

In a *gas*, the average distance between molecules is very large compared to that between the molecules of a solid or liquid. The enormous rapidity of diffusion of gases compared to that of liquids is suggestive of the fact that the molecules of a gas are in a state of rapid motion practically unaffected by the neighbouring ones. The molecules of a gas continually collide with one another and also with the sides of the containing vessel. At each collision the direction and magnitude of the velocity of a molecule is changed. When a molecule rebounds after collision with the wall of the containing vessel, its velocity is altered and it imparts a momentum to the wall of the container. This is the cause of the pressure exerted by a gas. Since a very small volume of the container contains an enormously large number of molecules, any appreciable area of the wall of the containing vessel receives an enormous number of impacts per second. The average number of impacts over the whole area of the vessel thus remains a constant. Hence, the pressure exerted by the gas remains a constant.

188. Assumptions about the Molecules of a Perfect Gas—

To deduce the gas laws two sets of assumptions are made. The first set is about the nature of the molecules of a perfect gas. These assumptions are—

(1) The molecules of a perfect gas do not exert any force on one another, so that they move along straight lines until they strike one another or on the wall of the containing vessel.

(2) The molecules are spherical in shape and obey the laws of elastic collision. The enclosing walls are supposed to be perfectly smooth and elastic.

(3) The volume occupied by a molecule is negligibly small compared to the total volume occupied by the gas. The molecules are considered as mass points.

The second set of assumptions are *probable* assumptions due to chaotic or random motion of the molecules of any gas, like the Brownian motion. These assumptions, due to *molecular chaos*, are made regarding (1) the distribution of molecules in the volume, (2) the direction of motion of the molecules, and (3) the distribution of velocities among the molecules.

(1) *Distribution of molecules in the volume*—It is supposed

that in absence of any external force the molecules are uniformly distributed over the volume at their disposal.

(2) *Direction of motion of the molecules*—In absence of any external force every direction of velocity is equally probable as every other direction ; *i.e.*, there is no unique direction in space so far as the velocity is concerned.

(3) *Distribution of velocities among the molecules*—Owing to collision all the molecules cannot have the same velocity but the velocity will vary between wide limits. It is assumed that not only all the molecules are uniformly distributed but the total number of molecules having a particular velocity are also uniformly distributed throughout the volume.

189. Boyle's Law—Consider a unit cube (a cube of unit length, breadth and height) containing n molecules of the gas each of mass m. Let a molecule moving with a velocity u in a direction perpendicular to two of the opposite faces of the cube, strike one of the two faces normally. The molecule rebounds with the same velocity u after impact on the wall in the opposite direction.

The momentum of the molecule before impact$=mu$ and its momentum after impact$=-mu$. Therefore, the change of momentum of the molecule or the momentum transferred to the face $=mu-(-mu)=2mu$.

If, for simplicity the molecule is supposed to move perpendicular to the two opposite faces of the cube without colliding with other molecules of the gas, it will strike each of the two faces of the cube $u/2$ times per second, for in one second, it moves a distance u and makes u collisions with the two opposite faces of the cube as the faces are separated by unit distance. Hence, the momentum transferred by the molecule on either of the two faces

$$=2mu\times\frac{u}{2}=mu^2$$

If $u_1, u_2 \ldots u_n$ represent the velocities of the n molecules, the momentum transferred by all the molecules per second on either of the two faces

$$=mu_1^2+mu_2^2+\ldots+mu_n^2$$

$$=m(u_1^2+u_2^2+\ldots+u_n^2)$$

$$=nm\bar{u}^2 \text{ where } \bar{u}^2=\frac{u_1^2+u_2^2+\ldots+u_n^2}{n}.$$

In fact, a molecule moves in any direction in space with any velocity c (say). If u, v and w represent the components of c perpendicular to the three sides of the vessel, evidently

$$c^2 = u^2 + v^2 + w^2.$$

If $\overline{c^2}$ represents the mean of all the values of c^2 for the various molecules and $\overline{u^2}$, $\overline{v^2}$ and $\overline{w^2}$, have similar significances, we shall have

$$\overline{c^2} = \overline{u^2} + \overline{v^2} + \overline{w^2}$$

The quantity $\overline{c^2}$ is called the **mean square velocity***.

Further, in absence of external force every direction of motion is equally probable ; also the molecules having a particular velocity are uniformly distributed. Hence, we have

$$\overline{u^2} = \overline{v^2} = \overline{w^2} = \tfrac{1}{3}\overline{c^2}.$$

Therefore, the momentum transferred by all the molecules to any of the faces of the cube per second

$$= nm\overline{u^2} = \tfrac{1}{3}nm\overline{c^2}.$$

By the second law of motion the momentum transferred per second is a measure of the force. Also the force applied per unit area is a measure of the pressure exerted. Since each face of the cube has unit area, the pressure p of the gas is given by

$$p = \tfrac{1}{3}nm\overline{c^2}.$$

Again, nm represents the mass of the n molecules contained in unit volume of the gas. Therefore nm is equal to the density ρ of the gas.

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*The mean square velocity $\overline{c^2}$ should be distinguished from the square of the mean velocity $\bar{c}^2$, or, the square root of the mean square velocities $\sqrt{\overline{c^2}}$ from the mean velocity $\bar{c}$. If 5 molecules move with velocities 1, 2, 3, 4 and 5 cms. per second, the mean square velcoity

$$\overline{c} = \frac{1^2+2^2+3^2+4^2+5^2}{5} = 11$$

The square-root of the mean square velocity $\sqrt{\overline{c^2}} = \sqrt{11} = 3{\cdot}32$ cms. per. sec. approximately.

The mean velocity $\bar{c} = \dfrac{1+2+3+4+5}{5} = 3$ cms. per. sec.

It can be shown that $\bar{c} = \dfrac{4}{\sqrt{6\pi}}\sqrt{\overline{c^2}} = 0{\cdot}92\sqrt{\overline{c^2}}$ approximately.

$\therefore\ p=\frac{1}{3}\rho\overline{c^2}$. This is known as *Bernouilli's equation*,*, as it was first obtained hy Daniel Bernouilli.

Replacing ρ by $\frac{1}{v}$ where v is the specific volume, $pv=\frac{1}{3}\overline{c^2}$.

Again, $\overline{c^2}$ is a constant for a particular temperature of the gas. Hence, for a particular temperature

$$Pv = \text{a constant.}$$

It will be observed that the possibility of collision of a molecule with another molecule while approaching a wall and its turning back has been ignored. This will not interfere with the result ; for, corresponding to one molecule that is turned *back*, there will be another which will be turned *forward* by collision. Since the number of molecules present is numerous, there will be a balance.†

This is the explanation of *Boyle's law from kinetic theory*.

Again the kinetic energy of translatory motion per unit volume of the gas $=\frac{1}{2}\rho\overline{c^2}$. Hence, the pressure of a gas is equal to two-thirds of the kinetic energy of translation of the gas molecules per unit volume.

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*The values of the root mean square of the velocity of molecules at normal temperature and pressure can be deduced from this equation.

Since $p=\frac{1}{3}\rho\overline{c^2}$, $\overline{c^2}=\frac{3p}{\rho}$ and $\sqrt{\overline{c^2}}=\sqrt{\frac{3p}{\rho}}$

The normal pressure $=76\times13{\cdot}6\times981=1{\cdot}016\times10^6$ dynes per sq. cm.

$\therefore\ \sqrt{\overline{c^2}}=\sqrt{\frac{3\times1{\cdot}016\times10^6}{}}$

The following table gives a list of molecular velocities at N. T. P.

Gas	Density at N. T. P. in gm. per c. c.	Root mean square velocity at N. T. P. in cms. per sec.	Mean velocity at N. T. P. in cms. per sec.
Hydrogen ...	$8{\cdot}9\times10^{-5}$	$18{\cdot}38\times10^4$	$16{\cdot}93\times10^4$
Oxygen ...	$14{\cdot}3\times10^{-4}$	$4{\cdot}61\times10^4$	$4{\cdot}25\times10^4$
Nitrogen ...	$12{\cdot}5\times10^{-4}$	$4{\cdot}93\times10^4$	$4{\cdot}54\times10^4$
Air ...	$12{\cdot}9\times10^{-4}$	$4{\cdot}85\times10^4$	$4{\cdot}47\times10^4$
Carbon dicxide ...	$19{\cdot}9\times10^{-4}$	$3{\cdot}93\times10^4$	$3{\cdot}62\times10^4$
Carbon monoxide...	$12{\cdot}5\times10^{-4}$	$4{\cdot}93\times10^4$	$4{\cdot}54\times10^4$

† The distance traversed by the molecule between two successive collisions is called the *free path*, the mean value of this for all the molecules is called *mean free path*.

190. Equation of State of a Perfect Gas—From the equation deduced in the last article, *viz.*, $pv=\frac{1}{3}\bar{c}^2$ has been obtained the well-known relation between the pressure, volume and temperature (absolute) of a perfect gas.

According to kinetic theory, the mean square velocity $\bar{c}^2$ is proportional to the absolute temperature of the gas.

∴ $pv=\frac{1}{3}\bar{c}^2=R'T$ where R' is the gas constant *per unit mass.*

$R'=\frac{R}{M}$ where R=the *gram-molecular* or *universal gas* constant and M, the molecular weight in gm.

$$\therefore\ pv=\frac{R}{M}T \text{ or, } pMv=RT$$

or, $pV=RT$ where $V=Mv$ the volume of a gram-molecule. This is the *equation of state of a perfect gas.*

The value of R can be determined from the fact that one gram-molecule of any gas occupies a volume of 22·4 litres at normal temperature and pressure. Thus,

$$R=\frac{pV}{T}=\frac{76\times 13{\cdot}6\times 981\times 22{\cdot}4\times 1000}{273}=8{\cdot}31\times 10^7$$

ergs. per degree

$$=\frac{8{\cdot}31\times 10^7}{4{\cdot}17\times 10^7}=1{\cdot}986 \text{ calories per degree}$$

(since $J=4{\cdot}17\times 10^7$ ergs per calorie).

191. Charles' Law and the Pressure-Temperature Law—The interpretation of the result of the foregoing article is that the volume of a certain mass of gas at constant pressure is proportional to its absolute temperature (Charle's law) or that the pressure of a certain mass gas at constant volume is proportional to its absolute temperature (Pressure-Temperature law).

Further, the result of the last article gives us a new definition of temperature of a gas. It is evident that it $T=0$, $\bar{c}^2=0$. Thus, at the absolute zero of temperature the average velocity and hence the kinetic energy of translation of the gas molecules vanishes.* The absolute temperature of a gas is thus a quantity which may be measured by the average kinetic energy of translation of the gas molecules. Temperature, according to kinetic theory, is thus a *statistical* conception.

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*The behaviour of matter at the absolute zero of temperature is almost inconceivable to us. It must not be imagined that a body at this point does not possess any energy at all though it does not possess any kinetic energy of translation. The energy content of a body at absolute zero is dealt with in Nernst's Heat Theorem.

192. Avogadro's Hypothesis—It has been observed in Art. 189 that the pressure p of a gas is given by $p=\frac{1}{3}nm\overline{c^2}$ where n is the number of molecules per unit volume, m is the mass of each molecule and $\overline{c^2}$ is mean square velocity of the molecules.

If two gases are at the same pressure, we evidently have

$$\tfrac{1}{3}n_1m_1\overline{c_1^2}=\tfrac{1}{3}n_2m_2\overline{c_2^2} \qquad \ldots \qquad \ldots \qquad (1)$$

where n_1 and n_2 represent the number of molecules of the two gases per unit volume, m_1 and m_2 the masses of each molecule of the two gases and $\overline{c_2^2}$ and $\overline{c_2^2}$ the mean square velocities of the molecules of the two gases.

Further, if two gases are at the same temperature, the mean kinetic energy of translation of the molecules of each gas is the same (Art. 191),

$$\therefore \quad \tfrac{1}{2}m_1\overline{c_1^2}=\tfrac{1}{2}m_2\overline{c_2^2} \qquad \ldots \qquad \ldots \qquad (2)$$

Dividing (1) by (2) we have

$$n_1=n_2$$

In other words, unit volume (or same volume) of all gases at the same temperature and pressure contains equal number of molecules.

193. Dalton's Law of Partial Pressures—If a certain volume of a mixture contains several gases of densities ρ_1, ρ_2, ρ_3, etc., having mean square velocities $\overline{c_1^2}$, $\overline{c_2^2}$, $\overline{c_3^2}$, etc., the pressure p exerted by the mixture will be given by

$$\begin{aligned} p &= \tfrac{1}{3}\rho_1\overline{c_1^2}+\tfrac{1}{3}\rho_2\overline{c_2^2}+\tfrac{1}{3}\rho_3\overline{c_3^2}+\text{etc.} \\ &= p_1+p_2+p_3+\text{etc.} \end{aligned}$$

where p_1, p_2, p_3, etc., are the pressures exerted separately by the constituent gases. In other words, the pressure exerted by a mixture of several gases is equal to the sum of the partial pressures of the constituent gases.

194. Graham's Law of Diffusion—Graham found that if two gases under the same temperature and pressure are allowed to diffuse into one another through fine pores, the rates of diffusion or the volumes of the two gases diffusing per second will be inversely proportional to the square-roots of their densities (Art. 172). It is evident that the rates of diffusion of two gases into one another will be directly proportional to their mean molecular velocities. In other words

$$\frac{\text{the rate of diffusion of a gas 1}}{\text{the rate of diffusion of a gas 2}}=\frac{\bar{c}_1}{\bar{c}_2}=\frac{\sqrt{\overline{c_1^2}}}{\sqrt{\overline{c_2^2}}} \quad \cdots \quad (1)$$

(see foot-note of Art. 183)

where $\overline{c_1^2}$ and $\overline{c_2^2}$ are the mean square velocities of the molecules of the gases 1 and 2 respectively.

Again, if the pressures of the two gases are the same,

$$\tfrac{1}{3}n_1m_1\overline{c_1^2}=\tfrac{1}{3}n_2m_2\overline{c_2^2} \quad \cdots\cdots \quad \cdots \quad (2)$$

where n_1 and n_2 are the number of molecules per unit volume and m_1 and m_2 are the masses of each molecule of the gases 1 and 2 respectively. From equation (2)

$\dfrac{\overline{c_1^2}}{\overline{c_2^2}}=\dfrac{n_2m_2}{n_1m_1}=\dfrac{\rho_2}{\rho_1}$ where ρ_1 and ρ_2 are the densities of the gases 1 and 2 respectively. Hence,

$$\frac{\sqrt{\overline{c_1^2}}}{\sqrt{\overline{c_2^2}}}=\frac{\sqrt{\rho_2}}{\sqrt{\rho_1}}$$

Therefore from equation (1)

$$\frac{\text{rate of diffusion of gas 1}}{\text{rate of diffusion of gas 2}}=\frac{\sqrt{\rho_2}}{\sqrt{\rho_1}}$$

195. Equipartition of Energy—It has been found in Art. 189 that the pressure exerted by a gas is given by

$$pv=\tfrac{1}{3}\overline{c^2}=R'T \quad \therefore \quad \overline{c^2}=3R'T$$

The mean kinetic energy of translation of a gas molecule

$$=\tfrac{1}{2}m\overline{c^2}=\tfrac{3}{2}mR'T \quad \cdots \quad \cdots\cdots \quad (1)$$

If the molecules are considered as mass points they possess only translatory energy. Each molecule has *three degrees of freedom, viz.*, three velocity co-ordinates to completely specify its directions of motion. To this class belong the *monatomic* molecules, *viz.*, those of the inert gases, such as argon, helium, etc., and the vapours of metals, such as mercury, sodium, potassium, etc. A degree of freedom is thus defined as *an independent mode in which a body may be displaced.* Boltzmann's "theorem on the equipartition of energy" states that *the total kinetic energy of a system is equally divided between all its degrees of freedom.* Hence, from above, the mean kinetic energy of translation of a monatomic gas molecule per degree of freedom at a temperature T (absol.)$=\tfrac{1}{2}mR'T=\tfrac{1}{2}kT$ where k is the gas constant *per molecule*, known as Boltzmann's constant.

If, on the other hand, the molecules of the gas are *diatomic*, such as those of hydrogen, oxygen, nitrogen, hydrochloric acid gas, air (which behaves like a diatomic gas), etc., the number of degrees of freedom of each molecule is five. For, such a molecule is supposed to consist of two atoms rigidly connected together possessing three degrees of freedom of *translatory* motion, and the two constituent atoms have two degrees of freedom for *rotational* motion. For more complex molecules, the number of degrees of freedom will be higher. For example, for *triatomic* molecules, such as those of corbon dioxide, sulphur dioxide, hydrogen sulphide, water vapour, etc., the number of degrees of freedom for each molecule is *six* and so on.

The principle of equipartition of energy states that the mean kinetic energy associated with each degree of freedom of translation or rotation of a gas molecule is of the same amount *viz.*, $\frac{1}{2}kT$ where T is the absolute temperature of the gas.

This theorem is further generalised. If instead of a chemically homogeneous gas, we have a mixture of several gases in equilibrium at a temperature, the mean kinetic energy per degree of freedom is the same for any of the gas molecules. Thus, if several gases are at the same temperature, the mean kinetic energy per degree of freedom of each molecule of the different gases is the same. It is also further shown that if a body consisting of a solid or liquid phase is in temperature equilibrium with a gas, the average kinetic energy per degree of freedom of a molecule of solid or liquid is the same as that of a gas molecule. The same kinetic energy is possessed by the solid or liquid if the gas is not present. Hence, if m_1, m_2, etc. represents the masses of each molecule of the different gases and R', R'', etc., the gas constants per unit mass of the different gases,

$$\tfrac{3}{2}m_1R'T=\tfrac{3}{2}m_2R''T=\text{etc.} \qquad \ldots \text{ from (1)}$$

where T is the common temperature of the gas

$$\text{or, } m_1R'=m_2R''=\text{etc.} \qquad \ldots \text{ from (2)}$$

Again, if M_1, M_2, etc., represent the molecular weights in gm. of the different gases, and R the universal (gram-molecular) gas constant,

$$R'=\frac{R}{M_1},\ R''=\frac{R}{M_2},\ \text{etc.}$$

$$\therefore \text{ from (2), } \frac{m_1R}{M_1}=\frac{m_2R}{M_2}=\text{etc.}$$

$$\text{or, } \frac{M_1}{m_1}=\frac{M_2}{m_2}=\ldots=N$$

where N represents the number of molecules within a gram-molecule and is the same for all gases. N is thus a universal constant for all gases and is known as the **Avogadro number.***

196. Classical Theory of Specific Heat of an Ideal Gas—The kinetic energy of a molecule per degree of freedom amounts to $\frac{1}{2}kT$ (Art. 195). Hence, the kinetic energy of a gram-molecule of the gas $\frac{f}{2}NkT$ (where f is the number of degrees of freedom of a molecule and N the Avogadro number). The total energy E of a gram-molecule of the gas can thus be written as

$E=\frac{f}{2}NkT+E_0$ (where E_0=energy other than kinetic). The gram-molecular specific heat at constant volume c_v is given by

$c_v=\left(\frac{dE}{dT}\right)_v=\frac{f}{2}Nk=\frac{f}{2}R$, where R is the universal gas-constant.

Also, if c_p represents the gram molecular specific heat at constant pressure

$$c_p-c_v=R$$

$$\therefore\quad c_p=c_v+R=\frac{f}{2}R+R=\frac{f+2}{2}R$$

$$\text{Hence, } \frac{c_p}{c_v}=\gamma \text{ (say)}=\frac{f+2}{2}R : \frac{f}{2}R=\frac{f+2}{f}$$

For monatomic gases, such as argon, helium, etc., and vapours of metals such as mercury, sodium, etc., $f=3$ (Art. 195). hence

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*The value of the Avogadro number N has been determined by several independent methods, the results of which are in excellent agreement within experimental errors. The simplest way of finding N is from electrolysis. From electrolysis we see that 0·01118 gram. of silver is deposited by one C. G. S. unit of electricity. Hence, the charge required to deposit one gram. atom (107·88 gms.) of silver=107·88÷·01118 or 9656 C. G. S. units. The charge e carried by one ion of silver is, according to Millikan. $4·775\times10^{-10}$ electrostatic or $1·59\times10^{-20}$ C. G. S. units. Hence N, the number of atoms in a gram atom of silver is given by $N=\frac{9656}{1·59\times10^{-20}}=6·066\times10^{23}$.

The value of N has been determined by Perrin (see *A* System of Physical Chemistry Vol. 1. McC Lewis, and by others.) The accepted value for N is $6·092\times10^{23}$.

The number of molecules in unit volume (1 c.c.) of a gas at normal pressure and temperature$=\frac{6·092\times10^{23}}{22·4\times10^{3}}$ or $2·75\times10^{19}$. This number is known as Loschmidt's number.

$\gamma=\frac{5}{3}=1{\cdot}66$. For diatomic gases, such as hydrogen, oxygen, etc., $f=5$, hence $\gamma=\frac{7}{5}=1{\cdot}40$. For triatomic gases, such as carbon dioxide, sulphur dioxide, hydrogen sulphide, etc., $f=6$, hence $\gamma=\frac{4}{3}=1{\cdot}33$ and so on.

197. Atomic Heat of Monatomic Solids : Dulong and Petit's Law—It has been observed in Art. 195 that the kinetic energy per degree of freedom of a molecule of a solid or liquid is the same as that of a gas, *viz.*, $\frac{1}{2}kT$. If we consider a monatomic solid, each molecule has three degrees of freedom of vibratory motion. Also, a gram-molecule of the solid contains N molecules, where N is the Avogadro number. Hence, the kinetic energy for one gram-molecule of the solid.

$=\frac{3}{2}NkT=\frac{3}{2}RT$, where R=the universal gas constant.

According to the kinetic theory, the molecules of a solid have no translatory motion but each molecule executes vibrations about some mean position (Art. 187). If the elastic force of restitution on a molecule is supposed to be proportional to its displacement from the mean position (*i.e.*, if the molecules are supposed to execute simple harmonic motions), it can be shown that the mean potential energy of a molecule is the same as its mean kinetic energy. Thus, the mean potential energy of a gram-molecule of the solid$=\frac{3}{2}RT$. Hence, the total energy E of a gram-molecule of a solid is given by

$$E=3RT+E_0 \text{ (where } E_0 \text{ is any other energy)}$$

Therefore, the gram-molecular specific heat

$$=\frac{dE}{dT}=3R=3\times 1{\cdot}986 \text{ or } 5{\cdot}958 \text{ calories per degree.}$$

This is Dulong and Petit's law of atomic heat for monatomic solids.

This law is found to be true in cases of solids, such as sodium, potassium, iron, magnesium, lead, etc. The law is found not to hold in cases, such as carbon, boron, silicon, etc., but these substances are found to obey the law at high temperatures. The deviations of the latter class of substances from Dulong and Petit's law at ordinary temperatures cannot be explained from the classical theory, *i. e.*, from the principle of equipartition of energy. The deviations, however, can be explained by the quantum theory formulated by Max Planck. Quantum theory shows that for any substance the atomic heat increases with temperature and attains above a *characteristic temperature* depending on the nature of the substance Dulong and Petit's value. The characteristic temperatures for the former class of substances are below ordinary temperature and hence they are found to obey Dulong and Petit's law at ordinary temperature, while the characteristic temperatures for the latter class are above ordinary temperature and they are found to deviate from Dulong and Petit's law at ordinary temperature but obey the law at higher than ordinary temperature.

Examples

1. Calculate the molecular velocity (*i.e.*, R. M. S. velocity) in the case of a gas whose density is 1·4 gm. per litre at a pressure of 76 cm. of mercury. Density of mercury=13·6 gms. per c.c. $g=981$ cms/sec^2 (Manchester B. Sc.)

The pressure p of a gas is related to the mean square velocity $\bar{c^2}$ by the equation

$p=\frac{1}{3}\rho\bar{c^2}$ where ρ=density of the gas (Art. 189)

$\therefore$ R. M. S. velocity $\sqrt{\bar{c^2}}=\sqrt{\frac{3p}{\rho}}=\sqrt{\frac{3\times 76\times 13{\cdot}6\times 981}{\cdot 0014}}$

$\log\sqrt{3} = \cdot 2386$

$\log\sqrt{76} = \cdot 9404$

$\log\sqrt{13{\cdot}6} = \cdot 5667$

$\log\sqrt{981} = 1{\cdot}4959$

$\log \text{Num.} = 3{\cdot}2416$

$\log \text{Num.} = 3{\cdot}2416$

$\log\sqrt{\cdot 0014} = \bar{2}{\cdot}5730$

$\log\sqrt{\bar{c^2}} = 4{\cdot}6686$

$\therefore \sqrt{\bar{c^2}} = 4{\cdot}56\times 10^4$ cms/sec.

2. Calculate the kinetic energy of hydrogen per gram molecule at $0°C$. (*C. U. 1951, Allahabad 1930, L. U. 1945*)

From equation (1), Art. 195 the kinetic energy (of translation) of a gas molecule$=\frac{3}{2}mR'T$ (m=mass of a molecule, R'=gas const. per unit mass).

$\therefore$ Mean K. E. per unit mass of hydrogen (putting $m=1$ gm)

$=\frac{3}{2}R'T$

=Hence, mean K. E. of a gram molecule of hydrogen

$=\frac{3}{2}MR'T=\frac{3}{2}RT$ (M=molecular wt. of hydrogen, R=universal gas const.)

$=\frac{3}{2}\times 2\times 4{\cdot}2\times 10^7\times 273$ ergs. ($T=273°$ Absol. $M=2$)

$=3{\cdot}36\times 10^{10}$ ergs.

Exercises

1. State clearly the fundamental assumptions of the kinetic theory of gases. Deduce from it Boyle's and Charles' laws for perfect gases. [*C. U. 1935*]

2. Deduce Boyle's law, Charles' law, and Avogardo's hypothesis for a perfect gas from the principles of the kinetic theory of gases. [*C. U. 1938*]

3. What is Dulong and Petit's law of specific heats for solids and gases? How has it been explained from the stand-point of the kinetic theory of gases? [*C. U. 1947*]

4. Deduce Boyle's law. Charles' law and Avogadro's hypothesis for a perfect gas from the principles of kinetic theory. [*C. U. 1947*]

5. Show that the pressure of a gas is equal to two-thirds of the kinetic energy of translation per unit volume. Calculate the kinetic energy of hydrogen per gram-molecule at 0°*C*. [*C. U. 1951 & L. U. 1948*]

6. What is the Kinetic theory view-point of gaseous systems? How would you relate the pressure, volume and temperature of an ideal gas from this view-point? [*C. U. 1952*]

7. Deduce an expression for the pressure of a gas from the view-point of kinetic theory, stating clearly the assumptions underlying your deduction [*C. U. 1956*]

8. Show how the general properties of the three states of matter are accounted for by the kinetic theory. Deduce the equation for a perfect gas. [*P. U. & E. P. U. 1952*]

9. Deduce an expression for the pressure of a gas on the kinetic theory. Find an expression for the root-mean square velocity of a gas molecule at a given temperature. [*N. U. 1954*]

10. What is kinetic theory of gases? Show that it leads directly to Boyle's law for a perfect gas, and that it indicates that the temperature of a gas is proportional to the mean kinetic energy of the molecules. [*P. U. B. E. P. U. 1943*]

11. Outline the essential features of the kinetic theory of gases. and derive the gas equation $Pv=RT$. [*L. U 1945*]

CHAPTER XV

UNITS AND DIMENSIONS

198. Units of Measurement—For the measurement of any physical quantity a suitable standard is chosen and the quantity in question is expressed in terms of the chosen standard. This standard is called the **unit**. Thus, for the measurement of a length, a standard called a *centimetre* is chosen, and the length to be measured is expressed by the number of times of centimetre which the length contains. For expressing a large length, a metre is chosen as the unit and for very short lengths, the millimetre is chosen as the unit.

199. Fundamental and Derived Units—The units of length, mass and time are called the *fundamental units* of measurement (Art. 4). In the centimetre, gramme, second system (C. G. S. system) of measurement the units of length, mass and time are a *centimetre*, a *gramme* and a *second* respectively. In the foot, pound, second system (F. P. S. system) of measurement the unit length, mass and time are a *foot*, a *pound* and a *second* respectively.

The units of all other physical quantities are derived from the fundamental units of measurement. Hence, these are called the *derived units*. Thus in the C. G. S. system, the unit of area is one *square centimetre*, the unit of velocity is one *centimetre per second*, the unit of force is *that force which acting on a mass of one gramme produces an acceleration of one centimetre per sec. per sec.* in it. In the F. P. S. system, the unit of area is one *square foot*, the unit of velocity is *one foot per second*, the unit of force is *that force which acting on a mass of one pound produces an acceleration of one foot per sec. per sec.* in it.

200. Dimensions—The dimensions of a physical quantity as has already been said (Chapter 1), represent an expression showing its relation to the fundamental units of measurement. The dimensions of length, mass and time are represented by L, M and T respectively. The dimensions of other physical quantities are expressed in terms of these.

Thus, the area of a rectangle is given by

$$\text{Area} = \text{length} \times \text{breadth}$$

Since the length and breadth both have the dimensions of length, the area is said to be of *two* dimensions in length. This is represented by

$$[\text{Area}] = [\text{length}]^2 = L^2$$

Similarly, volume=length × breadth × height and volume is of *three* dimensions in length which is represented by

$$[\text{Volume}] = [\text{length}]^3 = L^3.$$

In general, when a derived unit depends on the rth power of a fundamental unit, it is said to be of r dimensions in that fundamental.

Density is defined by the mass per unit volume

$$\therefore \quad [\text{Density}] = \frac{[\text{mass}]}{[\text{volume}]} = \frac{M}{L^3} = ML^{-3}$$

Specific gravity of a material is given by the ratio of the mass of a certain volume of the material to the mass of the same volume of some standard substance.

$$\therefore \quad [\text{Sp. gr.}] = \frac{[\text{mass}]}{[\text{mass}]} = \text{a } \textit{numerical quantity}$$

Hence, specific gravity has no dimensions.

Similarly, an angle in circular measure is given by the ratio arc divided by the radius. Thus, it is a ratio between two lengths. Hence, the circular measure of an angle has no dimensions.

Velocity of a moving body is defined by the length traversed per unit time. The dimensional equation of velocity is thus given by

$$[\text{Velocity}] = \frac{[\text{length}]}{[\text{time}]} = \frac{L}{T} = LT^{-1}$$

Acceleration is defined as the rate of change of velocity. Therefore, the dimensional equation of acceleration is

$$[\text{Acceleration}] = \frac{[\text{velocity}]}{[\text{time}]} = \frac{LT^{-1}}{T} = LT^{-2}$$

By the second law of motion the force on a body is measured by the product of the mass of the body into its acceleration.

$$[\text{Force}] = [\text{mass}] \times [\text{acceleration}] = MLT^{-2}$$

Momentum of a body is given by the product of the mass of the body into its velocity.

$$[\text{Momentum}] = [\text{mass}] \times [\text{velocity}] = MLT^{-1}$$

Impulse of a force over a certain time is measured by the product of the force and the time.

$$[\text{Impulse}] = [\text{force}] \times [\text{time}] = MLT^{-1}$$

Work done by a force is measured by the product of the force and the distance through which the point of application of the force move along its direction.

$$[\text{Work}] = [\text{force}] \times [\text{distance}] = ML^2T^{-2}$$

Since the energy of the body is measured by the work it is capable of doing, energy has same dimensions as work.

Pressure is defined by the force per unit area.

$$[\text{Pressure}] = \frac{[\text{force}]}{[\text{area}]} = \frac{MLT^{-2}}{L^2} = ML^{-1}T^{-2}$$

Stress is measured by the force on unit area. Hence, stress has the same dimensions as pressure, *viz.*, $ML^{-1}T^{-2}$. Strain in any of its forms is measured by the ratio of two quantities of the same nature. The longitudinal strain, lateral strain and shearing strain are measured by the ratio of two lengths, while the volume strain by the ratio of two volumes. Hence, strain is a numerical quantity having no dimensions. Poisson's ratio which is a ratio between lateral strain to longitudinal strain has no dimensions.

Modulus of elasticity (including Young's modulus, rigidity modulus and bulk modulus) is measured by the ratio of stress to strain. Hence,

[Modulus of elasticity]

$$= \frac{[\text{stress}]}{[\text{strain}]} = \frac{ML^{-1}T^{-2}}{[\text{a numerical quantity}]} = ML^{-1}T^{-2}.$$

Surface tension of a liquid is the force on unit length of the liquid boundary. Hence,

$$[\text{Surface tension}] = \frac{[\text{force}]}{[\text{length}]} = \frac{MLT^{-2}}{L} = MT^{-2}$$

Coefficient of viscosity is measured by the tangential stress per unit velocity gradient.

[Coefficient of viscosity]

$$= \frac{\dfrac{[\text{force}]}{[\text{area}]}}{\dfrac{[\text{velocity}]}{[\text{distance}]}} = \frac{[\text{force}] \times [\text{distance}]}{[\text{area}] \times [\text{velocity}]}$$

$$= \frac{MLT^{-2} \times L}{L^2 \times LT^{-1}} = ML^{-1}T^{-1}$$

Frequency of vibration is the number of vibrations executed per unit time. Hence,

$$[\text{Frequency}]=\frac{[\text{number}]}{[\text{time}]}=T^{-1}$$

201. Use of the Dimensional Equation—Dimensional equations are applied (1) for the conversion of the magnitude of a physical quantity from one system of measurement to any other system and (2) for studying the nature of dependence of a physical quantity with others upon which the former depends.

(*1*) *Conversion of the magnitude of a physical quantity from one system of measurement to another.*

(*a*) The surface tension of water is of magnitude 74 dynes per centimetre in the *C. G. S.* system of measurement. Calculate its magnitude in the *F. P. S.* system of measurement.

Let S be the magnitude of the surface tension in the *F. P. S.* system. Since the actual value of the surface tension is the same whatever be the units employed,

$$S[M_1T_1^{-2}]=74[MT^{-2}]$$

where M_1 and T_1 are the units of mass and time in the *F. P. S.* system M and T are the corresponding units in the *C. G. S.* system.

$$S=74\left[\frac{M}{M_1}\right]\left[\frac{T}{T_1}\right]^{-2}$$

$$=74\frac{1\text{ gramme}}{1\text{ pound}}\left[\frac{1\text{ sec.}}{1\text{ sec.}}\right]^{-2}$$

$$=74\frac{1}{454}=0{\cdot}163 \text{ poundals per foot approximately.}$$

(*b*) The acceleration of a falling body is 32 feet per second per second. Calculate its magnitude when the unit of length is a metre and that of time is a minute.

Let f be the magnitude of the acceleration required. Since the actual value of the acceleration is the same whatever be the units employed,

$$f[L_1T_1^{-2}]=32[LT^{-2}]$$

where L_1 and T_1 are the units of length and time respectively in the given system of measurement and L and T are the corresponding units in the *F. P. S.* system.

$$f=32\left[\frac{L}{L_1}\right]\left[\frac{T}{T_1}\right]^{-2}=32\left[\frac{L}{L_1}\right]\left[\frac{T_1}{T}\right]$$

$$=32\frac{1 \text{ foot}}{1 \text{ metre}}\left(\frac{1 \text{ minute}}{1 \text{ second}}\right)^2$$

$$=32\frac{12\times 2{\cdot}54}{100}.(60)^2=35113 \text{ metre per minute per minute.}$$

(2) *Nature of dependence of a physical quantity with others upon which the former depends.*

(*a*) The velocity of propagation of longitudinal waves through a gaseous medium depends upon the modulus of bulk elasticity and density of the medium. To study the nature of dependence, let the velocity be proportional to the αth power of the former and βth power of the latter, so that

$$v=kE^{\alpha}\rho^{\beta} \quad \ldots \quad \ldots \quad (1)$$

where v is the velocity of longitudinal waves, E and ρ are the bulk modulus and density of the medium and k is a numerical quantity. Putting down the dimensions of equation (1)

$$LT^{-1}=k(ML^{-1}T^{-2})^{\alpha}(ML^{-3})^{\beta}$$
$$=kM^{\alpha+\beta}L^{-\alpha-3\beta}T^{-2\alpha}$$

Since both the sides of equation (1) will have identical dimensions, equating the dimensions of both sides,

$$\left.\begin{aligned}\alpha+\beta&=0\\ -\alpha-3\beta&=1\\ -2\alpha&=-1\end{aligned}\right\} \quad \begin{aligned}&\text{Solving } \alpha=\tfrac{1}{2}\\ &\text{and } \beta=-\tfrac{1}{2}\end{aligned}$$

Substituting these values of α and β in equation (1)

$$v=kE^{\frac{1}{2}}\rho^{-\frac{1}{2}}$$
$$=k\sqrt{\frac{E}{\rho}}$$

The value of k can be obtained from an experimental determination of v and knowledge of E and ρ. The value of k is unity if adiabatic elasticity is taken for E.

(*b*) The rate of flow, *i.e.*, the volume of a liquid passing per second through a capillary tube depends upon the rate of drop of pressure per unit length of the tube, the radius of the tube and the coefficient of viscosity of the liquid. Thus, an expression for the rate of flow can be put down as

$$V=k\left(\frac{P}{l}\right)^{\alpha}r^{\beta}\eta^{\gamma} \quad \ldots\ldots \quad \ldots\ldots \quad (1)$$

where V, is the volume of liquid passing per second has a dimension

$$\frac{[\text{volume}]}{[\text{time}]}=L^3T^{-1}$$

$\frac{P}{l}$ is the pressure-drop per unit length and is of dimension

$$\frac{[\text{pressure}]}{[\text{length}]}=\frac{ML^{-1}T^{-2}}{L}=ML^{-2}T^{-2}$$

η, the coefficient of viscosity is of dimension $ML^{-1}T^{-1}$ and k is a numerical quantity.

Putting down the dimensions of both sides in (1)

$$L^3T^{-1}=k(ML^{-2}T^{-2})^{\alpha}L^{\beta}(ML^{-1}T^{-1})^{\gamma}$$

$$=kM^{\alpha+\gamma}L^{-2\alpha+\beta-\gamma}T^{-2\alpha-\gamma}$$

Equating the dimensions of both sides,

$$\left.\begin{aligned}\alpha+\gamma&=0\\-2\alpha+\beta-\gamma&=3\\-2\alpha-\gamma&=-1\end{aligned}\right\}\qquad\begin{aligned}\text{Solving, }\alpha&=1\\\gamma&=-1\\\text{and }\beta&=4\end{aligned}$$

Putting these values of α, β and γ in (1)

$$V=k\frac{P}{l}r^4\eta^{-1}$$

$$\text{or, }\quad V=k\frac{Pr^4}{l\eta}$$

The value of the quantity k can be found by experiment,

$k=\frac{\pi}{8}$(Art. 157).

(*c*) The resistance to the motion. of a sphere which moves through an infinite extent of viscous fluid depends upon r, the radius of the sphere ; η, coefficient of viscous fluid and v, the terminal velocity of the sphere.

Let the resisting force $F=kr^{\alpha}\eta^{\beta}v^{\gamma}$ .. (1)

Where k, α, β and γ are constants

Putting down the dimensions of both sides

$$MLT^{-2}=kL^{\alpha}(ML^{-1}T^{-1})^{\beta}(LT^{-1})^{\gamma}$$

$$=kL^{\alpha-\beta+\gamma}M^{\beta}T^{-\beta-\gamma}$$

Equating the dimensions of both sides,

$$\left.\begin{aligned}\beta&=1\\ \alpha-\beta+\gamma&=1\\ -\beta-\gamma&=-2\end{aligned}\right\}\quad \text{Solving}\quad \begin{aligned}\beta&=1\\ \gamma&=1\\ \alpha&=2\end{aligned}$$

Putting these values of α, β and γ in (1)
$F=k\,r\eta v$
Stokes showed that the value of $k=6\pi$
$\therefore\ F=6\pi r\eta v$ (Stokes' law)

202. Limitations of Dimensional Equations—(*a*) The application of dimensional equations to solution of problems is often useful but not always available. For instance, the force of attraction between two separate masses depends upon the attracting masses and the distance between them.

$$\text{Let } F=km^{\alpha}m'^{\beta}d^{\gamma}$$

where F is the force between masses m and m' separated by a distance d and k is a numerical quantity. Putting down the dimensions of both sides

$$MLT^{-2}=kM^{\alpha+\beta}L^{\gamma}$$

The above equation is not true as the left-hand side is of -2 dimensions in time whereas the right-hand side is of zero dimension in time. The reason for this break down is not due to any imperfection of the method but the constant k is not a purely numerical constant, the dimensions of which owing to our ignorance of the mechanism of gravitation are unintelligible.

(*b*) Let the space s described by a body starting with a velocity u and moving with a uniform acceleration f for a time t be given by

$$s=ku^{\alpha}f^{\beta}t^{\gamma}$$

Putting down the dimensions of both sides,

$$L=k(LT^{-1})^{\alpha}(LT^{-2})^{\beta}T^{\gamma}$$

$$=kL^{\alpha+\beta}T^{-\alpha-2\beta+\gamma}$$

Equating the dimensions of both sides,

$$\alpha+\beta=1$$

$$-\alpha-2\beta+\gamma=0$$

There are two equations which are not sufficient for finding the three quantities α, β and γ.

(*c*) The application of dimensions also fails when there are more than three possibilities of variation as the application of dimenstions cannot supply more than three equations. For example, on the dynamical theory of gases, the viscosity depends upon the mass of a molecule, the number of molecules per unit volume, their mean square velocity and effective radius. No definite result can be obtained without further knowledge of the physical circumstances.

However, although the application of dimensions is not always available for obtaining the nature of variation, it is often useful for checking the correctness of the variation as the terms on both sides must be of identical dimensions.

Exercise

1. Show from the dimensions of surface tension or otherwise that it is equal to the energy per unit area. [*C. U. 1916*]

2. Find the dimension of Young's modulus in terms of the fundamental units of length, mass and time. [*C. U. 1927*]

3. What do you mean by the dimension of a physical quantity in terms of mass, length and time?

Find the number of dynes in a force which acting on 15 kilograms for one minute produces a velocity of 8·6 kilometres per second. [*C. U. 1948*]

4. Explain what you mean by the dimension of a physical quantity. Calculate the dimensions of Young's modulus.

Assuming that the period of vibration of a tuning fork depends upon the the length of its prongs, and on the density and Young's modulus of the material, find by the method of dimensions a formula for the period of vibration. [*C. U. 1950*]

5. What do you understand by the dimensions of a physical quantity? Illustrate your answer by two examples.

Find the number of dynes in the force which, acting on 1 cwt. for one minute produces a velocity one mile per hour. Given 1 ft=30·5 cm. and 1 lb=453 gms. (*C. U. 1952*)

6. Find the dimensions of Young's modulus. [*C. U. 1953*]

7. What do you understand by the dimensions of a physical quantity? Derive by the use of the method of dimensions the expression for the time period of a simple pendulum.

The value of acceleration due to gravity is 32 ft. per sec-per sec. What will be its value if the unit of length is the mile and the unit of time is the minute. [*A. U. 1951, P. U. & E. P. U. 1952*]

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